THE TOPOLOGY OF SUBHARMONIC FUNCTIONS

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1. Introduction.

Let G be a ν-ply connected planar Jordan region whose boundary is denoted by B, ν being assumed to be a finite positive integer. Let $U=U(x,y)$ be a single-valued function pseudo-harmonic in G and continuous on B. If $U(x,y)$ has a finite number of points of relative extremum on B, then a relation

$$m - n = 2 - \nu$$

holds, where m is the number of boundary points affording relative minima to U and n is the sum of the orders of the saddle points of U on $\bar{G}$.

This theorem is a starting point of the theory introduced by M.Morse-M.H. Heins [1], in which the more general results under general assumptions has stated, but their methods highly depend upon the group-theoretic ones. Soon after Morse [1] has proved this relation by the most elementary method being able to consider as an extension of a previous paper of Morse-Van Schaack [1], in which they have concerned the so-called non-degenerate case only.

The object of the present paper is to extend the above mentioned relation to the subharmonic functions. We shall principally be interested to obtain the result. Therefore we shall begin with somewhat stronger assumptions than we need actually. In our case the so-called critical sets are not always the isolated ones (of course, not always the non-degenerate ones), and moreover they may consist of a critical line "en bloc". Difficulties will occur in this aspect. Thus we shall assume the stronger assumptions, some of which involve the essential parts of Morse's paper.

2. The basic assumptions.

Let G and B be the same as in the Morse's paper. Let $u=u(x,y)$ be a function defined on G, single-valued and subharmonic in G, and continuous on $\bar{G}$, and not reducing to a constant in any compact subregion of $\bar{G}$, where $\bar{G}=G+B$, where the subharmonicity of u means that the mean values of u in any small disc are always not less than the value at the center.

For the sake of simplicity, we shall confine ourselves to the case where u does not reduce to a constant on any subinterval of the boundary B unless the contrary is explicitly mentioned.

Definition 1. If a point (x, y) satisfies $u(x,y)=c$, then we call (x, y) lies on the level c. $\underline{U}(c)$ means the region below c, that is, the collection of all the points on $\bar{G}$ satisfying $u(x,y)<c$. Similarly, $\hat{U}(c)$, $\underline{U}(c)$, $\bar{U}(c)$ mean the region above c, the set below c, the set above c, respectively, which are defined by the collection of all the points satisfying $u(x,y)>c$, $u(x,y)\leq c$, $u(x,y)\geq c$ on $\bar{G}$, respectively.

Definition 2. Branching order of level at P with respect to a neighborhood $N(P)$. Let P be an arbitrary point of $\bar{G}$ and $N_\varepsilon(P)$ be a connected component of the set common to a fixed ε-neighborhood of P and to $\bar{G}$. Let $A_{N_\varepsilon}(P)$ be a set of the points each of which can be arcwisely connected to P along a continuous arc lying on the level $u(P)$ in $N_\varepsilon(P)$. That the level $u(P)$ at P has the finite branching order $b_{N_\varepsilon}(P)$ with respect to the neighborhood $N_\varepsilon(P)$ means that the point-set $A_{N_\varepsilon}(P)-P$ has a finite number of connected components. If it is not the case, we put $b_{N_\varepsilon}(P)=\infty$.

Theorem 1. At each point P of $\bar{G}$, $\lim_{\varepsilon\to 0} b_{N_\varepsilon}(P)$ exists and is either a finite non-negative integer or an infinite number.