ON DISTORTION IN SCHLICHT MAPPINGS

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0. Several distortion theorems have been derived, in various ways, for functions regular and schlicht in a circle. In the present Note we shall attempt certain estimations about their spherical derivative. The aim is to obtain estimates of spherical derivative, depending only on $r = |z|$, for family of functions regular, schlicht in the unit circle $|z| < 1$ and normalized at the origin. The results which will be obtained in the following lines are only partially precise. In fact, although the best possible bounds together with extremal functions can be found for points z comparatively near the origin, the precise bounds for remaining points are yet unknown. But it will be noteworthy to remark that the precise bounds are not analytic in the whole range of z.

On the other hand, the concept of spherical derivative is really rather useful for meromorphic functions than merely for regular functions. But, in comparison with rich results in the theory of schlicht functions regular in a circle, those referring to schlicht functions meromorphic in a circle are still poor. Making use of invariant character of spherical derivative with respect to any rotation of Riemann sphere, distortion inequalities will be derived for spherical derivative of certain schlicht functions meromorphic in a circle.

1. The spherical derivative of an analytic function $w(z)$ is defined as

$$(1.1)\qquad Dw(z) \equiv \frac{|w'(z)|}{1 + |w(z)|^2},$$

if z is a pole of the first order with residue c or of higher order, we put $Dw = 1/|c|$ or $Dw = 0$, respectively.

Consider first the family of functions $\{w(z)\}$ regular and schlicht in $|z| < 1$ and normalized at the origin such as

$$(1.2)\qquad w(0) = 0,\quad w'(0) = 1.$$

We shall attempt to estimate the spherical derivative of such functions from both sides. Now, as is well-known, the classical distortion theorems

$$(1.3)\qquad \frac{r}{(1+r)^2} \leqq |w(z)| \leqq \frac{r}{(1-r)^2},$$

$$(1.4)\qquad \frac{1-r}{1+r} \leqq \left|\frac{z\,w'(z)}{w(z)}\right| \leqq \frac{1+r}{1-r}.$$

due to Koebe-Bieberbach and to R. Nevanlinna respectively, hold good for any functions of the family. Moreover, for any z with $0 < r = |z| < 1$, the equality sign of left and right side is, in each case, realized only by Koebe's extremal function

$$(1.5)\qquad w = \frac{z}{(1+\varepsilon z)^2}\qquad (|\varepsilon| = 1),$$

and, in fact, merely at $z = \bar{\varepsilon}|z|$ and $z = -\bar{\varepsilon}|z|$, respectively.

Denoting now, for brevity, by

$$(1.6)\qquad r^* = \frac{\sqrt{5}-1}{2} = 0.618\ldots$$

the positive root of the quadratic equation $1 - r^2 = r$, we get, with regard to both bounds contained in Koebe-Bieberbach's distortion theorem (1.3), the relations

$$(1.7)\qquad \frac{r}{(1+r)^2} \leqq \frac{r}{(1-r)^2} \leqq 1\Big/\frac{r}{(1+r)^2}\qquad (r \leqq r^*),$$

$$(1.8)\qquad 1\Big/\frac{r}{(1-r)^2} < \frac{r}{(1+r)^2} < \frac{r}{(1-r)^2}\qquad (r^* < r < 1).$$

Hence, if $r = |z| \leqq r^*$, we have

$$\frac{r}{(1-r)^2} + \frac{(1-r)^2}{r} \leqq |w| + \frac{1}{|w|} \leqq \frac{r}{(1+r)^2} + \frac{(1+r)^2}{r},$$

or

$$(1.9)\qquad \frac{r^2+(1-r)^4}{r(1-r)^2} \leqq \frac{1+|w|^2}{|w|} \leqq \frac{r^2+(1+r)^4}{r(1+r)^2}\quad (r \leqq r^*).$$

Combining both relations (1.4) and (1.9), we obtain for spherical derivative which may be written in the form

$$Dw(z) = \left|\frac{w'}{w}\right| \frac{|w|}{1+|w|^2},$$

the following estimation:

$$(1.10)\qquad \frac{1-r^2}{r^2+(1+r)^4} \leqq Dw(z) \leqq \frac{1-r^2}{r^2+(1-r)^4}\qquad (r \leqq r^*)$$

The extremal functions for this distortion inequality must, as readily seen from the above argument, be of the form (1.5). For such a function the actual calculation shows that

$$w = \frac{z}{(1+\varepsilon z)^2},\qquad w' = \frac{1-\varepsilon z}{(1+\varepsilon z)^3};$$

$$(1.11)\qquad Dw = \frac{|w'|}{1+|w|^2} = \frac{|1-\varepsilon^2 z^2|}{|z|^2+|1+\varepsilon z|^4},$$

and hence the left and right bound in (1.10) is indeed attained at $z = r\bar{\varepsilon}$ and $z = -r\bar{\varepsilon}$, and only at these points, respectively.

We note here, in passing, that the same is valid for distortion inequality