ON ROBIN'S CONSTANT AND A DISTORTION THEOREM.

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1. Let $\mathcal{B}$ be any domain whose Green function be denoted by $G(z,\zeta)$. The Robin constant $\gamma(\zeta)$ of $\mathcal{B}$ with respect to the pole ζ is defined by the relation

$$(1.1)\quad G(z,\zeta)+\lg|z-\zeta|=\gamma(\zeta)+\tau(z,\zeta),$$

the residual term satisfying the limit equation

$$\tau(z,\zeta)=O(|z-\zeta|)\quad\text{as}\quad z\to\zeta.$$

If the pole coincides with the point at infinity, then $|z-\zeta|$ has only to be replaced by $|z|^{-1}$. Although, in the following lines, we shall suppose, for the sake of brevity, that ζ is a finite point, the argument will easily be modified for the case $\zeta=\infty$.

Now, making use of Hadamard's variational method, Bergman(1) has recently shown that the residual term in (1.1) satisfies the inequality

$$(1.2)\quad \tau(z,\zeta)+\tau(\zeta,z)\geqq 0,$$

which, by the symmetry property of Green function, can also be expressed in the form

$$(1.3)\quad 2\left(G(z,\zeta)+\lg|z-\zeta|\right)\geqq\gamma(z)+\gamma(\zeta)$$

Moreover, he has also noticed that, in the special case of a simply-connected domain $\mathcal{B}$, this result contains, as an immediate consequence, a classical distortion theorem due to Löwner stating that

$$(1.4)\quad |f'(w)|\leqq\frac{|w|^2}{|w|^2-1}$$

is valid for any function $f(w)$ schlicht in $|w|>1$ and normalized at the point at infinity such as

$$(1.5)\quad f(w)=w+\sum_{\nu=0}^{\infty}c_\nu w^{-\nu}\quad(|w|>1).$$

2. Now, if we restrict ourselves to simply-connected domains, the inequality (1.3) of question can conversely be deduced from Löwner's distortion formula (1.4) also by an elementary procedure, and moreover the extremal domains for the estimation (1.3) can be explicitly be determined. In the present Note, these facts will be established.

Let $z=\varphi(w)$ be a function mapping $|w|>1$ onto $\mathcal{B}$. Let ω be any point with $1<|\omega|<\infty$, and suppose temporarily $|\varphi(\bar\omega)|\neq 1$. Then

$$(2.1)\quad \Phi(w)=\varphi'(\bar\omega)\frac{|\omega|^2-1}{|\varphi(\bar\omega)|^2-1}\,\frac{\overline{\varphi(\bar\omega)}\,\varphi\left(\frac{\bar\omega w-1}{w-\omega}\right)-1}{\varphi\left(\frac{\bar\omega w-1}{w-\omega}\right)-\varphi(\bar\omega)}$$

is a function schlicht in $|w|>1$, normalized at $w=\infty$ such that $\Phi(\infty)=\infty$ and $\Phi'(\infty)=1$, and whose derivative is given by the expression

$$\Phi'(w)=\varphi'(\bar\omega)\frac{(|\omega|^2-1)^2}{(w-\omega)^2}\,\frac{\varphi'\left(\frac{\bar\omega w-1}{w-\omega}\right)}{\left(\varphi\left(\frac{\bar\omega w-1}{w-\omega}\right)-\varphi(\bar\omega)\right)^2}\quad(|w|>1).$$

If we now put

$$\frac{\bar\omega w-1}{w-\omega}=W,\qquad \bar\omega=\Omega,$$

then $|W|>1$, $|\Omega|>1$ and the above expression becomes

$$(2.2)\quad \Phi'\left(\frac{\bar\Omega W-1}{W-\Omega}\right)=\varphi'(\Omega)\,\varphi'(W)\left(\frac{W-\Omega}{\varphi(W)-\varphi(\Omega)}\right)^2.$$

Applying here the Löwner's distortion theorem, we get

$$(2.3)\quad \left|\Phi'\left(\frac{\bar\Omega W-1}{W-\Omega}\right)\right|\leqq\frac{1}{1-\left|\frac{W-\Omega}{\bar\Omega W-1}\right|^2}=\frac{|\bar\Omega W-1|^2}{(|\Omega|^2-1)(|W|^2-1)}.$$

Hence, from (2.2), we obtain

$$(2.4)\quad |\varphi'(\Omega)\varphi'(W)|\leqq\frac{|\bar\Omega W-1|^2}{(|\Omega|^2-1)(|W|^2-1)}\left|\frac{\varphi(W)-\varphi(\Omega)}{W-\Omega}\right|^2.$$

Remembering the continuity character, we see that the last inequality holds good for any Ω and W with moduli larger than unity without the restriction $|\varphi'(\Omega)|\neq 1$. We call here, by the way, attention to the fact that if $\mathcal{B}$ is a domain containing the point at infinity and possessing the reduced modulus equal to unity and if the mapping function is normalized such as $\varphi(\infty)=\infty$ and $\varphi'(\infty)=1$, then the Löwner's distortion formula $|\varphi'(W)|\leqq|W|^2/(|W|^2-1)$ is reproduced from (2.4) as a limiting case $\Omega\to\infty$.

Denoting now by $w=\psi(z)$ the inverse function of $z=\varphi(w)$, the Green function of $\mathcal{B}$ is given by

$$(2.5)\quad G(z,\zeta)=\lg\left|\frac{\overline{\psi(\zeta)}\,\psi(z)-1}{\psi(z)-\psi(\zeta)}\right|.$$

Hence, the Robin constant is, by definition (1.1), expressed in the form

$$(2.6)\quad \gamma(\zeta)=\lim_{z\to\zeta}\left(G(z,\zeta)+\lg|z-\zeta|\right)=\lg\frac{|\psi(\zeta)|^2-1}{|\psi'(\zeta)|}.$$