ON LACUNARY NON-HARMONIC TRIGONOMETRIC SERIES

By Masatomo UDAGAWA

(Communicated by T. Kawata)

I. The following theorem of Zygmund on lacunary trigonometric series is well known.[1)]

A. If $n_{k+1}/n_k > \lambda > 1$, n_k's being integers, and the series $\sum(a_k^2+b_k^2)$ converges, then

$$(1) \quad \sum_{k=1}^{\infty}(a_k \cos n_k x + b_k \sin n_k x)$$

is the Fourier series of a function $f(x)$ belonging to the class L^r, r being any positive number and

$$\left\{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^r dx\right\}^{1/r} \leqq A_{r\lambda}\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2}$$

where $A_{r\lambda}$ depends only on r and λ.

B. Under the conditions of A, we have

$$B_{r\lambda}\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2} \leqq \left\{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^r dx\right\}^{1/r},$$

where $B_{r\lambda}$ is positive and depends only on r, and λ.

We shall prove that a theorem of same type is valid, even if the integral character of the numbers is not assumed.

Theorem 1. If $\lambda_{k+1}/\lambda_k \geqq \lambda > 1$, $\lambda_1 \geqq 1$ (this is not on essential restriction, so for our convention we suppose this in the following all theorems), and if the series $\sum_{k=1}^{\infty}|c_k|^2$ converges, then

$$(2) \quad \sum_{k=1}^{\infty} c_k e^{i\lambda_k x}$$

is almost everywhere convergent to a function $f(x)$ belonging to every $L^{r,\sigma}(-\infty,\infty)$, $L^{r,\sigma}$ means the class of functions whose r th power in absolute values are integrable with respect to a monotone function $\sigma(x)$,

$$\sigma(x) = \frac{1}{\pi}\int_{-\infty}^{x}\frac{1-\cos t}{t^2}dt, \text{ and}$$

$$(3) \quad \left\{\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x)\right\}^{1/r} \leqq A'_{r\lambda}\left\{\sum_{k=1}^{\infty}|c_k|^2\right\}^{1/2}$$

holds, where $A'_{r\lambda}$ depends only on r and λ.[4)]

Theorem 2. Under the conditions of Theorem 1.

$$(4) \quad B'_{r\lambda}\left\{\sum_{k=1}^{\infty}|c_k|^2\right\}^{1/2} \leqq \left\{\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x)\right\}^{1/r}, \quad r \geqq 1$$

where $B'_{r\lambda}$ is a positive constant which depends only on r and λ.

Theorem 3. Under the conditions of Theorem 1, the series (2) converges in the mean with exponent r over every finite interval to $f(x)$.

We shall prove more strong results than Theorem 1, that is;

Theorem 4. If the conditions of Theorem 1 are satisfied, and

$$S^*(x) = \sup_n |S_n(x)|,$$

where

$$S_n(x) = \sum_{k=1}^{n} c_k e^{i\lambda_k x}$$

then

$$(5) \quad \int_{-\infty}^{\infty}|S^*(x)|^r d\sigma(x) \leqq D_{r,\lambda}\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x) \quad r > 1$$

where $D_{r,\lambda}$ depends only on r and λ.

Theorem 5. Under the conditions of Theorem 4, we have

$$\int_a^b |S^*(x)|^r dx \leqq D\int_a^b |f(x)|^r dx, \quad r > 1$$

where D depends on r, λ, a and b.

Theorem 6. There exists a constant $\mu\ (>0)$ which depends only $\sum|c_k|^2$ and λ such that the function $e^{\mu|f(x)|}$ belonging to the class $L^{1,\sigma}(-\infty,\infty)$.

Remark. All the above theorems still hold even if we take $\sigma_h(x)$ $(0<h\leqq 1)$ for $\sigma(x)$, where

$$\sigma_h(x) = \frac{1}{\pi}\int_{-\infty}^{x}\frac{\sin^2 ht}{ht^2}dt$$

2. Proof of Theorem 1. The almost everywhere convergence are already proved by M.Kac,[2)] and so we shall prove the inequality (3).

By Holder's inequality, if $r < r'$, we have

$$\left\{\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x)\right\}^{1/r} \leqq \left\{\int_{-\infty}^{\infty}|f(x)|^{r'} d\sigma(x)\right\}^{1/r'},$$

and hence it is sufficient to consider $r = 2m$, $m = 1, 2, \cdots$.

Moreover it is well known that if we prove the inequality