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SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH TURNING POINTS AND SINGULARITIES I

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§ 1. Introduction.

Differential equations containing a positive small parameter ε

$$\varepsilon^2 \frac{d^2y}{dx^2} - p(x)y = 0 \tag{1.1}$$

are considered. The independent variable x is complex. The coefficient $p(x)$ is a rational function of the form $p(x) = r(x)/q(x)$, where $r(x)$ and $q(x)$ have no common factors. Zeros of $r(x)$ are called turning points and zeros of $q(x)$ (and possibily the point at infinity) are singular points of the differential equation (1.1).

When the parameter ε tends to zero, asymptotic solutions of (1.1) are valid only in some domain in the x-plane. The principal parts of the formal solutions are of the form

$$y(x, \varepsilon) \sim p(x)^{-1/4} \exp\left[\pm \frac{1}{\varepsilon} \int p(x)^{1/2} dx\right] \qquad (\varepsilon \to 0), \tag{1.2}$$

which are called WKB approximations. In this paper we consider several cases of $p(x)$ and show how to construct unbounded domains called canonical regions. The validity of WKB approximations in the canonical region can be established as in Evgrafov-Fedoryuk [1] or Nakano [3].

In § 3 and § 5 the case $r(x) = (x-1)^2$, $q(x) = x$ is treated and in § 4 and § 6 the case $r(x) = -(x-1)^2$, $q(x) = x$ is treated. These two cases are very similar but some difference appears between them. The two cases have similar property in the small but different property in the large. In § 2 some common property between them is treated. In § 7 canonical paths are treated. In these cases $x=1$ is a turning point of order two, the origin is a regular singular point and the point at infinity is an irregular singular point of (1.1). In § 8 the case $r(x) = (x-1)^2$, $q(x) = x^3$ is treated, which is the simplest case of (1.1) having a turning point at $x=1$ and two irregular singular points at $x=0$ and point at infinity. In the part 2 we shall consider cases containing a logarthmic term in (2.1) below and a matching method.

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