SOME PROPERTIES OF ASYMPTOTIC DISTRIBUTIONS

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This note contains two different problems. In §1, §2 we shall give some results similar as the one which were obtained by Kac and Steinhaus.(1) The definitions used here of asymptotic distributions are different from them, and the hypothesis in the theorem are less restrictive. In §3, we are concerned with some limit theorems.

§1.

Definition 1. Let $x(t)$ be a measurable function of a real variable t taking values in R^n.
For every $T>0$, we define

$$\varphi_T(E)=\frac{1}{2T}\, m E_t\left[-T\le t\le T,\ x(t)\in E\right],$$

E being an arbitrary Borel set in R^n.
Then $\varphi_T(E)$ is a distribution function for every fixed T.
If the distribution function φ_T tends to a distribution function φ(2) for $T\to\infty$, we say that $x(t)$ has an asymptotic distribution function φ. This definition is due to Harteman and Wintner.
Now we shall prove the theorem.

Theorem 1. If $x(t)$ has an asymptotic distribution function, then for any continuous function $f(x)$ in R^n, $f(x(t))$ has an asymptotic distribution function.

To prove the theorem we need a following lemma.

Lemma 1. A measurable function has an asymptotic distribution function φ, if and only if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{iu\cdot x(t)}\,dt$$

exists, uniformly in $|u|\le C$ for any $C>0$.

This lemma is known.(3)

Proof of Theorem. For simplicity, we restrict ourselves for the case where $x(t)$ is a real function. We write

$$f(x(t)) = y(t)$$

By Lemma 1, it is sufficient to prove that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{iu\cdot y(t)}\,dt$$

exists, uniformly in $|u|\le C$ for every $C>0$. Obviously

$$\frac{1}{2T}\int_{-T}^{T} e^{iu\cdot y(t)}\,dt=\int_{-\infty}^{\infty} e^{iu\cdot f(x)}\,d\varphi_T(x),$$

so that it suffices to show that

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} e^{iu\cdot f(x)}\,d\varphi_T(x)$$

exists, uniformly in $|u|\le C$.

Since $\varphi_T\to\varphi$, for arbitrarily small $\varepsilon>0$, we can choose x_0 and T_0 which satisfy following conditions:

i) $-x_0, x_0$ are continuity points of φ.

ii) $\{1-\varphi_T(x_0)\}+\varphi_T(-x_0)<\varepsilon$ for all $T>T_0$.

Then we have

$$\left|\left(\int_{-\infty}^{-x_0}+\int_{x_0}^{\infty}\right)e^{iu\cdot f(x)}\,d\varphi_T(x)\right|<\varepsilon,\quad (T>T_0)$$

$$\left|\left(\int_{-\infty}^{-x_0}+\int_{x_0}^{\infty}\right)e^{iu\cdot f(x)}\,d\varphi(x)\right|<\varepsilon.$$

Now, we can choose $F(x)$ which is absolutely continuous in $(-x_0, x_0)$ such that

$$|f(x)-F(x)|<\frac{\varepsilon}{C},\quad \text{for } -x_0\le x\le x_0.$$

Then

$$\left|\int_{-x_0}^{x_0}\left\{e^{iu\cdot f(x)}-e^{iu\cdot F(x)}\right\}d\varphi_T(x)\right|\le\int_{-x_0}^{x_0}|u|\,|f(x)-F(x)|\cdot d\varphi_T(x)<\varepsilon.$$

Similarily

$$\left|\int_{-x_0}^{x_0}\left\{e^{iuf(x)}-e^{iu\cdot F(x)}\right\}d\varphi(x)\right|<\varepsilon.$$

Therefore we have

$$\left|\int_{-x_0}^{x_0}e^{iu\cdot f(x)}\,d\{\varphi_T(x)-\varphi(x)\}\right|\le\left|\int_{-x_0}^{x_0}\left\{e^{iuf(x)}-e^{iuF(x)}\right\}d\varphi_T(x)\right|$$
$$+\left|\int_{-x_0}^{x_0}\left\{e^{iu\cdot f(x)}-e^{iuF(x)}\right\}d\varphi(x)\right|+\left|\int_{-x_0}^{x_0}e^{iu\cdot F(x)}\,d\{\varphi(x)-\varphi_T(x)\}\right|$$
$$\le 2\varepsilon+\left|\int_{-x_0}^{x_0}e^{iu\cdot F(x)}\,d\{\varphi(x)-\varphi_T(x)\}\right|.$$

But the integration by parts shows that

$$\left|\int_{-x_0}^{x_0}e^{iu\cdot F(x)}\,d\{\varphi(x)-\varphi_T(x)\}\right|\le\left|\left[e^{iuF(x)}\{\varphi(x)-\varphi_T(x)\}\right]_{-x_0}^{x_0}\right|$$
$$+\left|\int_{-x_0}^{x_0}\{\varphi(x)-\varphi_T(x)\}\,d\left(e^{iuF(x)}\right)\right|$$
$$\le 2\varepsilon+\left|\int_{-x_0}^{x_0}uF'(x)\{\varphi(x)-\varphi_T(x)\}e^{iuF(x)}\,dx\right|.$$