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LOCALIZATION OF THE COEFFICIENT THEOREM

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Abstract

Let f be holomorphic and univalent in $D = \{|z| < 1\}$ and set $K(z) = z/(1-z)^2$. We prove $|f^{(n)}(z)/f'(z)| \le K^{(n)}(|z|)/K'(|z|)$ at each $z \in D$ and for each $n \ge 2$. This inequality at $z = 0$ is just the coefficient theorem of de Branges, the very solution of the Bieberbach conjecture. The equality condition is given in detail. In the specified case where $f(D)$ is convex we have again a similar and sharp result. We also consider $|f^{(n)}(z)/f'(z)|$ for f univalent in a hyperbolic domain Ω with the Poincaré density $P_\Omega(z)$ and the radius of univalency $\rho_\Omega(z)$ at $z \in \Omega$. We obtain the estimate $(\rho_\Omega(z)/P_\Omega(z))^{n-1}|f^{(n)}(z)/f'(z)| \le n!4^{n-1}$ at $z \in \Omega$ for $n \ge 2$, together with the detailed equality condition on f, Ω, and z.

1. Introduction

Let $\mathcal{U}$ be the family of functions holomorphic and univalent in $D = \{z; |z| < 1\}$. Writing $f_\gamma(z) = \bar{\gamma} f(\gamma z)$ for $f \in \mathcal{U}$ and for $\gamma \in \partial D \equiv \{z; |z| = 1\}$, we know that important members of $\mathcal{U}$ are K_γ, the γ-rotations of the Koebe function $K(z) = z/(1-z)^2$. The coefficient theorem proved by L. de Branges [B] then reads as follows. For each $f \in \mathcal{U}$ and for each $n \ge 2$, the inequality

$$\left|\frac{f^{(n)}(0)}{f'(0)}\right| \le n!n \tag{1.1}$$

holds. If the equality holds in (1.1) for an $n \ge 2$, then $f = f'(0)K_\gamma + f(0)$ for some $\gamma \in \partial D$. Conversely the equality holds in (1.1) for all $n \ge 2$ and for all $f = AK_\gamma + B$, where $A \ne 0, B$, and $\gamma \in \partial D$ are complex constants.

By induction we have

$$K_\gamma^{(n)}(z) \equiv (K_\gamma)^{(n)}(z) = \frac{\gamma^{n-1} n!(n + \gamma z)}{(1 - \gamma z)^{n+2}} \quad (n \ge 1, \gamma \in \partial D), \tag{1.2}$$

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