H. YAMAGUCHI
KODAI MATH. J.
19 (1996), 117—136

# ORTHOGONAL DECOMPOSITION RELATED TO MAGNETIC FIELD, AND GRUNSKY INEQUALITY

HIROSHI YAMAGUCHI

## 1. Introduction

Let $D$ be a bounded domain in $\boldsymbol{R}^3$ with $C^\omega$ smooth boundary surfaces $\Sigma$. Let $\sigma=adx+bdy+cdz$ be a $C^\infty$ closed 1-form on $\bar{D}$ $(=D\cup\Sigma)$. By putting $\tilde{\sigma}=\sigma$ in $\bar{D}$ and $=0$ outside $D$, we consider the usual Weyl's orthogonal decomposition: $\tilde{\sigma}=*\omega+dF$ in $\boldsymbol{R}^3$, where $\omega$ is a $L^2$ closed 2-form in $\boldsymbol{R}^3$ and $dF\in\mathrm{Cl}[dC_0^\infty(\boldsymbol{R}^3)]$.

In §4 we shall show that $\omega$ is a harmonic 2-form in $\boldsymbol{R}^3\backslash\Sigma$ of the form $\omega=dp$ and that $p$ and $F$ are written into the following integral formulas:

$$p(x)=\Big(\frac{1}{4\pi}\int_\Sigma\frac{(a,\,b,\,c)\times\boldsymbol{n}_y}{\|x-y\|}dS_y\Big)\cdot dx \qquad \text{for } x\in\boldsymbol{R}^3,$$

$$F(x)=\frac{1}{4\pi}\int_\Sigma\frac{(a,\,b,\,c)\cdot\boldsymbol{n}_y}{\|x-y\|}dS_y-\frac{1}{4\pi}\int_D\frac{\operatorname{div}(a,\,b,\,c)}{\|x-y\|}dv_y \qquad \text{for } x\in\boldsymbol{R}^3,$$

where $\boldsymbol{n}_y$ is the unit outer normal vector of $\Sigma$ at $y$, $dx=(dx,\,dy,\,dz)$, and $\cdot$ means the formal inner product.

In §2 we briefly recall the definition of surface current densities on $\Sigma$ and their properties studied in [6]. In §3 we shall prove an approximation lemma concerning improper integrals. This lemma is not only useful to prove the above integral formulas but also to show the fact that $\omega$ is related to the magnetic field. Precisely, if we write $\omega=\alpha dy\wedge dz+\beta dz\wedge dx+\gamma dx\wedge dy$ and define $B=(\alpha,\,\beta,\,\gamma)$ in $\boldsymbol{R}^3\backslash\Sigma$, then $B$ is a magnetic field induced by a surface current density $JdS_x$ on $\Sigma$ such that $B$ is the strong limit of a sequence of usual magnetic fields $\{B_n\}_n$ in $\boldsymbol{R}^3$: $\lim_{n\to\infty}\int_{\boldsymbol{R}^3}\|B_n(x)-B(x)\|^2dv_x=0$. In §5 we shall show that this fact implies the existence of equilibrium current densities $\mathcal{J}dS_x$ on $\Sigma$. The notion of equilibrium current densities were introduced in [6] motivated by the electric solenoid.

In §6 the integral formulas in $\boldsymbol{R}^3$ stated above is extended into those in the complex $z$-plane. We then obtain a new proof of Grunsky inequality (cf. [4]), which implies a necessary and sufficient condition for the case when the inequality is reduced to equality. It gives us many examples of such cases.

Received October 25, 1994; revised May 8, 1995.