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ON THE FUNCTIONAL EQUATION $f^n=e^{P_1}+\cdots+e^{P_m}$ AND RIGIDITY THEOREMS FOR HOLOMORPHIC CURVES

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Introduction and statement of results

For each positive integer N we set

$$E_N=\{e^{P_1}+\cdots+e^{P_m}\ \ P_j\in \boldsymbol{C}[z],\ \deg P_j\leqq N0=1,\ -\ ,\ m),\ m\in \boldsymbol{N}\}.$$

In 1929 J. F. Ritt [4] showed the following theorem.

THEOREM A. *Let $g_0, g_1, \cdots, g_n$ be elements of E_1 and f be a holomorphic function on $\{z\,;\,\omega_1<\arg z<\omega_2\}$ $(\omega_2-\omega_1>\pi)$ satisfying $g_n f^n+g_{n-1}f^{n-1}+\cdots+g_0=0$. Then $f\in E_1$.*

It seems to be natural to ask whether Theorem A is valid with E_1 replaced by E_N $(N\geqq 2)$. However, if $g_n\neq 1$, the function $f(z)=\sin(\pi z^2)/\sin \pi z$ gives a negative answer to the above question.

Let $g:\boldsymbol{C}\to \boldsymbol{P}_m$ be a holomorphic curve of finite order, $D_0, D_1, \cdots > D_{m-1}$ be hyperplanes and D_m be a hypersurfac of degree n $(\geqq 2)$ satisfying $D_0\cap\cdot\cap D_m-\emptyset$, $g(\boldsymbol{C})\cap(D_0\cup\cdots\cup D_m)=\emptyset$. We ask whether the image of g is contained in the intersection of hypersurfaces of $\boldsymbol{P}_m$. This problem is related to the functional equation $f^n+g_{n-1}f^{n-1}+\cdots+g_0=0$ $(g_0, \cdots, g_{n-1}\in E_N)$ for an entire function /. M. Green [1] treated the first non-trivial case $f^2=e^{2\varphi_1}+e^{2\varphi_2}+e^{2\varphi_3}$ $(\varphi_1, \varphi_2, \varphi_3\in \boldsymbol{C}[z])$ and showed that / is a linear combination of e^{φ_1}, e^{φ_2}, e^{φ_3}. He also showed that, if $g:\boldsymbol{C}\to\boldsymbol{P}_2$ is a holomorphic curve of finite order omitting the two lines $\{Z_0=0\}$ and $\{Z_1=0\}$ and the conic $\{Z_0^2+Z_1^2+Z_2^2=0\}$, then the image of g lies in a line or a conic ([1]).

In this paper we shall show the following results.

THEOREM 1. *Let $P_1, \cdots, P_m$ be polynomials, $N=\max_j \deg P_j$, $N\geqq 2$, $A_j=P_j^{(N)}(0)/N!(j=1,\cdots, m)$, n $(\geqq 2)$ be an integer and f be a holomorphic function on $\{z\,;\,\omega_1<\arg z<\omega_2\}$ $(\omega_2-\omega_1>\pi/N)$. Assume that $\#\{j\ A_j=v,\ j=1, \cdots, m\}=1$ for every vertex v of the convex hull of $\{A_j\}_{j=1}^m$, and that $f^n=e^{P_1}+\cdots+e^{P_m}$ on $\{z\,;\,\omega_1<\arg z<\omega_2\}$. Then f is an element of E_N.*

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