A. YAMADA
KODAI MATH. J.
11 (1988), 317—324

BOUNDED ANALYTIC FUNCTIONS AND METRICS OF CONSTANT CURVATURE ON RIEMANN SURFACES

BY AKIRA YAMADA

1. Introduction.

Let $\boldsymbol{B}(\Omega)$ be the set of bounded analytic functions $f: \Omega\rightarrow\Delta$, where Ω is a simply connected hyperbolic Riemann surface and Δ is the unit disc. Let $\boldsymbol{B}_0(\Omega)$ be the set of locally schlicht functions belonging to $\boldsymbol{B}(\Omega)$. The Poincaré metric λ_Ω of the surface Ω has constant curvature $\equiv -4$. Explicitly, we have $\lambda_\Delta(z)=|dz|/(1-|z|^2)$. It is well known that the pull-back $f^*\lambda_\Delta(z)=|f'(z)||dz|/(1-|f(z)|^2)$ of λ_Δ via $f\in\boldsymbol{B}_0(\Omega)$ (resp. $f\in B(\Omega)$) is a metric of constant curvature $\equiv -4$ regular (resp. with isolated singularities) on Ω. The main result of this paper asserts that the converse of the above relation holds. For simplicity, we denote by $M(X)$ the set of C^∞ conformal metrics of constant curvature $\equiv -4$ on a Riemann surface X.

THEOREM 1. *For all $\lambda\in M(\Omega)$, there exists an $f\in\boldsymbol{B}_0(\Omega)$ such that $\lambda=f^*\lambda_\Delta$. Moreover, $\lambda=g^*\lambda_\Delta$ for $g\in\boldsymbol{B}_0(\Omega)$ if and only if g is of the form $g=\gamma\circ f$ where $\gamma\in$*Möb(Δ), *the set of Möbius transformations leaving Δ fixed.*

THEOREM 2. *Let E be an arbitrary closed discrete subset of Ω. If $\lambda\in M(\Omega\setminus E)$ has non-positive integral indices at every point in E, then there exists an $f\in\boldsymbol{B}(\Omega)$ such that $\lambda=f^*\lambda_\Delta$ on $\Omega\setminus E$. Moreover, $\lambda=g^*\lambda_\Delta$ on $\Omega\setminus E$ if and only if g is of the form $g=\gamma\circ f$ with $\gamma\in$*Möb(Δ).

Theorems 1 and 2 show that the set of metrics of constant negative curvature is in a one-to-one correspondence with the set of bounded analytic functions in Ω modulo Möb(Δ). Theorem 2 is an improvement of Theorem 29.1 in Heins [2] and many of the results concerning metrics of constant curvature in [2] are easy consequence of Theorem 1. Also, Theorem 1 allows us to define the monodromy homomorphism χ of a metric of constant curvature. Theorem 6 answers the question when the image of the homomorphism χ acts discontinuously on Δ. In the last section, we prove a theorem which shows that a theorem in [5] is false.

The author wishes to thank the referee for helpful comments and advice.

Received December 8, 1987