N. WATANABE
KODAI MATH. J
9 (1986), 165—169

NONCOMMUTATIVE EXTENSION OF AN INTEGRAL REPRESENTATION THEOREM OF ENTROPY

Dedicated to Professor H. Umegaki on his 60th birthday

BY NOBORU WATANABE

Introduction

In 1964, Umegaki proved a theorem of McMillan type concerning the integral representation of entropy in the measure theoretic framework, about which we briefly review in § 1. Noncommutative probability theory is important to analyse some physical systems [1, 2, 4, 5, 6, 7, 10, 11, 12, 13, 16, 17]. In this paper, using various results obtained in operator algebras, we extend this theorem to that for noncommutative systems.

§ 1. Integral representation of entropy

Let X be a compact metric space and $\mathfrak{B}(X)$ be the σ-field of all Borel sets in X. We denote a homeomorphism on X by T and the set of all T-invariant regular probability measures $p, q, \cdots$ on X by P_T. Let $\mathcal{P}$ be a finite partition of X and we put $\mathfrak{M}_n=\bigvee_{k=1}^{n} T^{-k}\mathcal{P}$ and $\mathfrak{M}_\infty=\bigvee_{k=1}^{\infty}\mathfrak{M}_k$. Then the entropy of each $p\in P_T$ is defined by

$$S(p)=-\lim\frac{1}{n}\Sigma_U p(U)\log p(U) \qquad (n\to\infty),$$

where Σ_U means the summation over U of the atomic sets in $\mathcal{P}\vee\mathfrak{M}_{n-1}$. For any $p\in P_T$, we denote the conditional probability functions of $U\in\mathcal{P}$ with respect to $\mathfrak{M}_n$ and $\mathfrak{M}_\infty$ by $P_p(U|\mathfrak{M}_n)$ and $P_p(U|\mathfrak{M}_\infty)$ respectively. Now we define the $\mathfrak{M}_\infty$-measurable function $h_p(x)$ on X as follows:

$$h_p(x)=-\sum_{U\in\mathcal{P}} P_p(U|\mathfrak{M}_\infty)\log P_p(U|\mathfrak{M}_\infty)(x) \qquad p\text{-a. e.} \quad x\in X,$$

for any $p\in P_T$. Then, the next important theorem [14] of McMillan type holds.

THEOREM 1. *For any finite partition $\mathcal{P}$, there universally exists a Borel measurable function $h(x)$ on X such that it is bounded, non-negative, T-invariant and satisfies*

(1) $h(x)=h_p(x)$ *p-a. e.* $x\in X$ *and for every* $p\in P_T$,

Received June 20, 1985