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KODAI MATH. J.
7 (1984), 208–210

ON STRONG NORMALITIES

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In the paper [1], the author asks for an example in a complete K-metric space where K is a strongly normal cone of a reflexive infinite dimensional Banach space. Our main purpose is to present such an example.

Let V be a normed space. A set $K \subset V$ is said to be a cone if and only if

(1) K is closed;

(2) If $u, v \in K$, then $au+bv \in K$ for all $a, b \geqq 0$;

(3) $K \cap (-K)=\{\theta\}$ where θ is the zero of the space V, and

(4) $K^0=\emptyset$ where K^0 is the interior of K.

We say $u \geqq v$ if and only if $u-v \in K$. The cone K is said to be strongly normal if there is $c>0$ such that if $z=\sum_{i=1}^{n} b_i x_i$, $x_i \in K$, $\|x_i\|=1$, $\sum_{i=1}^{n} b_i=1$, $b_i \geqq 0$ implies $\|z\|>c$. The mapping $\phi: K \to K$ is said to be lower semicontinuous if $\{u_n\}$ and $\{\phi(u_n)\}$ are both weakly convergent, then $\lim \phi(u_n) \geqq \phi(\lim u_n)$. In a finite dimensional space, the weak topology and the strong topology are same, but, in an infinite dimensional space, they are different. Therefore if we can get an example in a complete K-metric space where K is a strongly normal cone of a reflexive infinite dimensional Banach space, the above definition of the lower semicontinuity will be more significant; we also generalize the value of K-metric $d(x, y)$ to an infinite dimensional space and improve [1, 2].

From now on, we assume that $(V, \langle\cdot, \cdot\rangle)$ is an inner product space over R (all real numbers). $\langle\cdot, \cdot\rangle$ is an inner product on V, and $\|x\|=\langle x, x\rangle^{1/2}$, $x \in V$.

LEMMA 1 (Parallelogram Identity [4]). *Let V be an inner product space over R. Then*

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \qquad (x, y \in V).$$

LEMMA 2 (Polarization Identity [4]). *Let V be an inner product space over R. Then*

$$\langle x, y\rangle=\left\|\frac{x+y}{2}\right\|^2-\left\|\frac{x-y}{2}\right\|^2 \qquad (x, y \in V).$$

Remark. Let $0<c<1$. From Lemma 1, if $\|x-y\| \leqq c$, $\|x\|=1$, and $\|y\|=1$,

Received May 25, 1983