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Face algebras I—A generalization of quantum group theory

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§ 0. Introduction

In the last decade, it was recognized that monoidal categories or categories with tensor product play essential roles in many branches of mathematics and mathematical physics such as Jones' index theory, low-dimensional topology and conformal field theory. For example, Ocneanu's classification theory of II_1-subfactors deeply depends on the structure of the monoidal categories of bimodules over von Neumann algebras (cf. [**O**]). On the other hand, it is known that a certain class of monoidal additive categories gives rise to 2-dimensional topological quantum field theories (see Turaev [**T**]).

These developments naturally stimulate to construct non-trivial examples of monoidal categories. Many examples are constructed using representation theory of bialgebras (quantum groups). However, there still exist monoidal categories which have no representation-theoretic interpretation; for example, II_1-subfactors of type D–E have no representation-theoretic counterparts.

In this paper, we begin to study a new algebraic structure named face algebra, which is a generalization of bialgebra. Although, its definition is much more complicated than that of bialgebra, the category of its (co-)modules still has the structure of monoidal abelian category. By considering additional structures on it (such as an antipode, a universal R-matrix, a ribbon structure and a $*$-structure), we obtain monoidal categories with rich additional structures.

In this paper, we concentrate our attention on elementary properties of face algebras and their (co-)module categories. Non-trivial examples and applications will be given elsewhere (cf. [**H1–6**]).

In Sections 1, 2 and 3, We show several basic formulas for face algebras, their antipodes and their universal R-matrices. In Sections 4 and 5, we show that the categories of modules and comodules of face algebras naturally become monoidal categories. When face algebras have antipodes or universal R-matrices, we also discuss the rigidity or the braiding structure of these categories.

Throughout this paper, we work on a fixed ground field $\boldsymbol{K}$. For an algebra A, we denote its product by $m = m_A$. For a coalgebra C, we denote its coproduct and its counit by $\Delta = \Delta_C$ and $\varepsilon = \varepsilon_C$ respectively. We also use the "sigma" notation $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ $(a \in C)$ (cf. [**S**]).