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Buekenhout geometries of rank 3 which involve the Petersen graph

By Thomas MEIXNER

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1. Introduction.

Consider the diagram $(c^{*}\cdot P)$: $\circ\overset{c}{\text{——}}\circ\overset{P}{\text{——}}\circ$.

Here, the symbol $\circ\overset{c}{\text{——}}\circ$ stands for *the circle geometry with* 4 *points* and $\circ\overset{P}{\text{——}}\circ$ for the geometry of *the Petersen graph.* We will determine all simply connected geometries $\underline{G}$ with this diagram and flag-transitive automorphism group. It will turn out that there are exactly five simply connected ones. One of them is related to the alternating group of degree 6, two of them are related to the Mathieu groups M_{11} and M_{12}, all these three are finite.

One is related to the symmetric group of degree 9 and to the sporadic group He, and one to the group $SO_5(5)$, and we do not know, whether they are finite or infinite.

To be precise, we will prove the following theorem.

THEOREM. *Let $\underline{G}$ be a connected, simply connected geometry with diagram $(c^{*}\cdot P)$ and flag-transitive automorphism group G. Then $\underline{G}$ is one of the geometries $\underline{G}_5$, $\underline{G}_6$, $\underline{G}_9$, $\underline{G}_{11}$ or $\underline{G}_{12}$ defined in the next section, and G is isomorphic to one of the groups G_{6a}, G_{6b} (if $\underline{G}$ is $\underline{G}_6$), or G_{11} (if $\underline{G}$ is G_{11}), or G_{12a}, G_{12b}, G_{12c}, G_{12d}, G_{12e} (if $\underline{G}$ is $\underline{G}_{12}$), or G_{9a}, G_{9b} (if $\underline{G}$ is $\underline{G}_9$), or G_{5a}, G_{5b} (if $\underline{G}$ is $\underline{G}_5$), all defined in the last section, respectively.*

Here, $\underline{G}_{11}$ is a geometry with 66 points and automorphism group $G_{11}=M_{11}$, $\underline{G}_{12}$ is a geometry with 4752 points and automorphism group G_{12d} (resp. G_{12e})$=(A_4\times M_{12})2$ and projects onto a geometry for M_{12}, $\underline{G}_6$ is a geometry with 6480 points and automorphism group $G_{6b}=3(A_6\times A_6)2$, while we do not know, whether the geometries $\underline{G}_5$ and $\underline{G}_9$ are finite or infinite. They project onto finite geometries for $SO_5(5)$ and Σ_9 respectively and have automorphism groups G_{5b} and G_{9b} respectively.

The remark on the automorphism groups of the examples is almost trivial: the pairs $(\underline{G}, \mathrm{Aut}(\underline{G}))$ have to appear in the list, hence one has only to check in every case, which of the groups acting on the same geometry $\underline{G}$ is the "biggest one".