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The $W^{k,p}$-continuity of wave operators for Schrödinger operators

Dedicated to Professor S. T. Kuroda on his sixtieth birthday

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1. Introduction, Theorems.

For the pair of Schrödinger operators $H_0 = D_1^2 + \cdots + D_m^2$ and $H = H_0 + V$, where $D_j = -i\partial/\partial x_j$, $j=1, \cdots, m$, and V is the multiplication operator with the real valued function $V(x)$, the wave operators $W_\pm = W_\pm(H, H_0)$ are defined by

$$W_\pm = s-\lim_{t\to\pm\infty} e^{itH} e^{-itH_0}, \tag{1.1}$$

where $s-$indicates the strong limit in $L^2(\boldsymbol{R}^m)$. In this paper, we prove under suitable conditions on $V(x)$ that $W_\pm$ are bounded in the Sobolev spaces $W^{k,p}(\boldsymbol{R}^m)$ for any $1 \leqq p \leqq \infty$ and $k=0, 1, \cdots, l$. The merit of the wave operators is that they intertwine the part H_c of H on the continuous spectral subspace $L_c^2(H)$ and H_0: $H_c = W_\pm H_0 W_\pm^*$ on $L_c^2(H)$. Hence the $W^{k,p}(\boldsymbol{R}^m)$-boundedness of $W_\pm$ implies that the functions $f(H_0)$ and $f(H)P_c(H)$, $P_c(H)$ being the orthogonal projection onto $L_c^2(H)$, have equivalent operator norms from $W^{k,p}(\boldsymbol{R}^m)$ to $W^{k',q}(\boldsymbol{R}^m)$ for any $1 \leqq p$, $q \leqq \infty$ and $k, k'=0, 1, \cdots, l$:

$$\begin{aligned} C_1 \| f(H_0) \|_{B(W^{k,p}, W^{k',q})} &\leqq \| f(H) P_c(H) \|_{B(W^{k,p}, W^{k',q})} \\ &\leqq C_2 \| f(H_0) \|_{B(W^{k,p}, W^{k',q})}, \end{aligned} \tag{1.2}$$

where the constants are independent of f. We shall apply (1.2) to obtain, among others, the $L^p - L^q$ estimates for the propagators of the time dependent Schrödinger equations $i\partial u/\partial t = Hu$ and of the wave or Klein-Gordon equations with potentials $\partial^2 u/\partial t^2 + Hu + \mu^2 u = 0$, and the "Fourier multiplier theorems" for the generalized eigenfunction expansions associated with H.

We assume that $V(x)$ satisfies the following assumption, where $\mathcal{F}$ is the Fourier transform, $\langle x \rangle = (1+|x|^2)^{1/2}$, $l \geqq 0$ is a fixed integer, and $m_* = (m-1) \cdot/(m-2)$. For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_m)$, $D^\alpha = D_1^{\alpha_1} \cdots D_m^{\alpha_m}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

ASSUMPTION 1.1. *$V(x)$ is a real valued function on $\boldsymbol{R}^m$, $m \geqq 3$, such that for any $|\alpha| \leqq l$ $\mathcal{F}(\langle x \rangle^\sigma D^\alpha V) \in L^{m_*}(\boldsymbol{R}^m)$ for some $\sigma > 2/m_*$ and satisfies one of the*

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