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On the structure of the moduli space of harmonic eigenmaps

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1. Introduction and preliminaries.

It is well-known that a map $F: M\rightarrow S^N$ of a Riemannian manifold M into the Euclidean N-sphere $S^N\subset \boldsymbol{R}^{N+1}$ is *harmonic* [3] iff the induced vector-valued function $F: M\rightarrow \boldsymbol{R}^{N+1}$ satisfies the equation

$$\Delta^M F=\mu F\,, \tag{1}$$

where Δ^M is the Laplacian on M and the scalar μ is uniquely determined by F, in fact, μ is nothing but the energy density $e(F)=\mathrm{trace}\,\|F_*\|^2$ of F. (We work in the C^∞-category, i.e., we assume that all manifolds, maps, bundles etc. are of class C^∞.)

Applying an $(n+1)\times(N+1)$-matrix $A\in M(n+1, N+1)$ to both sides of (1) we infer that $f=A\cdot F$ defines a harmonic map of M into S^n of energy density $e(f)=\mu(=e(F))$ provided that A maps the image of F into S^n. In this case, we say that f is *derived* from F and write $f\leftharpoonup F$. Define a symmetric endomorphism $\langle f\rangle$ of $\boldsymbol{R}^{N+1}$ by $\langle f\rangle=A^{\top}A-I_{N+1}$. The condition $|f|^2=1$ is equivalent to that $\langle f\rangle$ is perpendicular to $\mathrm{proj}\,[F(x)]$ for all $x\in M$ with respect to the usual inner product $\langle C, C'\rangle=\mathrm{trace}\,(C'^{\top}\cdot C)$, $C, C'\in S^2(\boldsymbol{R}^{N+1})$. Clearly $\langle f\rangle$ depends only on the equivalence class of f, where two maps $f', f'': M\rightarrow S^n$ (derived from F) are said to be *equivalent* if $f''=U\cdot f'$ for some $U\in O(n+1)$. Restricting ourselves from here on to full maps (i.e., assuming that the image always spans the range) we obtain that, given a full harmonic map $F: M\rightarrow S^N$, the equivalence classes of full harmonic maps $f: M\rightarrow S^n$ that are derived from F can be parametrized (via $f\rightarrow\langle f\rangle$) by the convex body

$$\mathcal{L}_F=\{C\in\mathcal{E}_F\,|\,C+I_{N+1}\geqq 0\} \tag{2}$$

('$\geqq$' stands for positive semidefinite), where

$$\mathcal{E}_F=(\mathrm{span}\{\mathrm{proj}\,[F(x)]\,|\,x\in M\})^{\perp}\subset S^2(\boldsymbol{R}^{N+1})\,. \tag{3}$$