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Hausdorff dimensions of self-similar sets and shortest path metrics

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Introduction.

Let (X, d) be a complete metric space and let $f_1, f_2, \cdots, f_N$ be contractions from X to itself, that is,

$$\sup_{x, y \in X} \frac{d(f_i(x), f_i(y))}{d(x, y)} < 1 .$$

Then it follows that

THEOREM 1.1 (Hutchinson [**Hu**]). *There exists a unique non-empty compact set K such that*

$$K = \bigcup_{i=1}^{N} f_i(K) .$$

K is called a self-similar set with respect to $((X, d), \{f_i\}_{i=1}^{N})$.

This paper contains two main subjects. First in § 2, we will study the Hausdorff dimension of a self-similar set. For the case that X is an Euclidian space $\boldsymbol{R}^n$ and f_i's are similitudes, there is a well-known result by Moran [**M**].

THEOREM 1.2. *Let $X=\boldsymbol{R}^n$ and let f_i be an r_i-similitude of $\boldsymbol{R}^n$ for $i=1, 2, \cdots, N$; that is, for all $x, y \in \boldsymbol{R}^n$,*

$$d(f_i(x), f_i(y)) = r_i d(x, y) ,$$

where d is the ordinary Euclidean distance on $\boldsymbol{R}^n$. If there exists an open set $O \subset \boldsymbol{R}^n$ such that

$$\bigcup_{i=1}^{N} f_i(O) \subset O \quad \text{and} \quad f_i(O) \cap f_j(O) = \emptyset \qquad \text{for } i \neq j ,$$

then the Hausdorff dimension of the self-similar set (K, d) with respect to $((\boldsymbol{R}^n, d), \{f_i\}_{i=1}^{N})$ is the unique number α that satisfies

$$\sum_{i=1}^{N} r_i^{\alpha} = 1 . \tag{1.1}$$