J. Math. Soc. Japan
Vol. 47, No. 2, 1995

Standard generalized vectors for algebras of unbounded operators

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(Received Apr. 22, 1993)
(Revised Aug. 11, 1993)

§1. Introduction.

In [7, 8] we defined the notion of generalized vectors for O^*-algebras which is a generalization of cyclic vectors to study the structure of O^*-algebras. In particular, we defined and studied the notion of (full) standard generalized vectors which makes it possible to develop the Tomita-Takesaki theory in O^*-algebras. In this paper we shall continue such a study for O^*-algebras (called generalized von Neumann algebra) which are an unbounded generalization of von Neumann algebras. Let λ be a standard generalized vector for a generalized von Neumann algebra $\mathcal{M}$ on $\mathcal{D}$. Then it has been shown in [7] that a one-parameter group $\{\sigma_t^\lambda\}_{t\in\boldsymbol{R}}$ of $*$-automorphisms of $\mathcal{M}$ is defined and λ satisfies the KMS-condition with respect to $\{\sigma_t^\lambda\}$. We shall show a Radon-Nikodym type property which establishes a link between the modular automorphism groups $\{\sigma_t^\lambda\}$ and $\{\sigma_t^\mu\}$ of $\mathcal{M}$ for two full standard generalized vectors λ and μ: There uniquely exists a strongly continuous map $t\in\boldsymbol{R}\rightarrow U_t\in\mathcal{M}_u\equiv\{U\in\mathcal{M}\,;\,\bar{U}$ is unitary$\}$ such that $U_{t+s}=U_t\sigma_s^\lambda(U_s)$ and $\sigma_t^\mu(X)=U_t\sigma_t^\lambda(X)U_t^*$ for all $s, t\in\boldsymbol{R}$ and $X\in\mathcal{M}$. The map $t\in\boldsymbol{R}\rightarrow U_t\in\mathcal{M}$ is called the cocycle associated with μ with respect to λ and denoted by $[D\mu : D\lambda]$. Further, we shall show that $\{[D\mu : D\lambda]_t\}_{t\in\boldsymbol{R}}$ is a one-parameter group if and only if the domain $D(\mu)$ of μ is $\{\sigma_t^\lambda\}$-invariant and $\|\mu(\sigma_t^\lambda(X))\|=\|\mu(X)\|$ for all $X\in D(\mu)$ and $t\in\boldsymbol{R}$ if and only if $\{[D\mu : D\lambda]_t\}\subset\mathcal{M}_b^{\sigma^\lambda}$ $\equiv\{A\in\mathcal{M}\,;\,\bar{A}$ is bounded and $\sigma_t^\lambda(A)=A$ for all $t\in\boldsymbol{R}\}$. Then, we say that μ commutes with λ. These results are generalization of the Connes cocycle theorem [1, 17] for von Neumann algebras. We shall extend the Pedersen-Takesaki Radon-Nikodym theorem [12, 17] for von Neumann algebras to generalized von Neumann algebras. Let λ be a full standard generalized vector for $\mathcal{M}$ and $\mathcal{M}_\eta^{\sigma^\lambda}$ the set of all non-singular positive self-adjoint operators A in $\mathcal{H}$ satisfying $\{E_A(t)\,;\,-\infty<t<\infty\}''\lceil\mathcal{D}\subset\mathcal{M}_b^{\sigma^\lambda}$, where $\{E_A(t)\}$ is the spectral resolutions of A. For every $A\in\mathcal{M}_\eta^{\sigma^\lambda}$ we can define a standard generalized vector λ_A for $\mathcal{M}$ satisfying $\sigma_t^{\lambda_A}(X)=A^{2it}\sigma_t^\lambda(X)A^{-2it}$ and $[D(\lambda_A)_\sigma : D\lambda]_t=A^{2it}\lceil\mathcal{D}$ for all $X\in\mathcal{M}$ and $t\in\boldsymbol{R}$, and so $(\lambda_A)_\sigma$ commutes with λ, where $(\lambda_A)_\sigma$ is the full extension of λ_A. Con-