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Posinormal operators

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0. Introduction.

In this paper we study the properties of a large subclass of $\mathcal{B}(\mathcal{H})$, the set of all bounded linear operators $T:\mathcal{H}\rightarrow\mathcal{H}$ on a Hilbert space $\mathcal{H}$. As is customary, we refer to $T^{*}T-TT^{*}$ as the *self-commutator of* T, denoted $[T^{*}, T]$. A self-adjoint operator P is *positive* if $\langle Pf, f\rangle\geqq 0$ for all $f\in\mathcal{H}$; the operator T is *normal* if $[T^{*}, T]=0$ and T is *hyponormal* if $[T^{*}, T]$ is positive. When T^{*} is hyponormal, we say T is *cohyponormal*; T is *seminormal* if T is hyponormal or cohyponormal. If T is the restriction of a normal operator to an invariant subspace, then T is *subnormal*.

If $A\in\mathcal{B}(\mathcal{H})$ is to belong to our subclass, then A must not be "too far" from normal; more precisely, there must exist an *interrupter* $S\in\mathcal{B}(\mathcal{H})$ such that $AA^{*}=A^{*}SA$, or equivalently, $[A^{*}, A]=A^{*}(I-S)A$. Two observations suggest the additional requirement that S be self-adjoint, even positive: (1) since AA^{*} is self-adjoint, each operator A in our subclass must satisfy $A^{*}S^{*}A=A^{*}SA$; (2) since $\langle SAf, Af\rangle=\langle A^{*}SAf, f\rangle=\|A^{*}f\|^{2}$ for all f, the interrupter S must be positive on Ran A (the range of A).

DEFINITION. If $A\in\mathcal{B}(\mathcal{H})$, then A is *posinormal* if there exists a positive operator $P\in\mathcal{B}(\mathcal{H})$ such that $AA^{*}=A^{*}PA$. $\mathcal{P}(\mathcal{H})$ will denote the set of all posinormal operators on $\mathcal{H}$. A is *coposinormal* if A^{*} is posinormal.

We note that if A is posinormal with interrupter P and V is an isometry (that is, $V^{*}V=I$), then, as one can easily check, VAV^{*} is posinormal with interrupter VPV^{*}. Consequently, posinormality is a unitary invariant (that is, if A is posinormal and T is unitarily equivalent to A, then T is also posinormal).

If the posinormal operator A is nonzero, the associated interrupter P must satisfy the condition $\|P\|\geqq 1$ since $\|A\|^{2}=\|AA^{*}\|=\|A^{*}PA\|\leqq\|A^{*}\|\|P\|\|A\|=\|P\|\|A\|^{2}$. We will make repeated use $\sqrt{P}$, whose existence is guaranteed by the functional calculus for (positive) self-adjoint operators. P need not be unique, as we will soon see; the following result gives a sufficient condition for the uniqueness of P.