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Bounds on the fundamental group of a manifold with almost nonnegative Ricci curvature

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Several recent papers have given generalizations of the following classical theorems relating the curvature of a compact Riemannian manifold to its fundamental group:

THEOREM 1 (Bochner). *Let M be a compact Riemannian manifold. If M has Ricci curvature nonnegative everywhere and strictly positive somewhere, then* $H^1(M; R)=0$.

THEOREM 2 (Myers). *A complete Riemannian manifold with Ricci curvature bounded from below by a positive constant is compact and has finite fundamental group.*

THEOREM 3 (Milnor). *The fundamental group of a compact Riemannian manifold with nonnegative Ricci curvature has polynomial growth.*

For example, Berard [**2, 3**] and Elworthy-Rosenberg [**5**] extended Bochner's theorem to the case where the Ricci curvature is arbitrarily negative, as long as the set where Ricci is negative has small enough volume; we refer to such a set as a deep well of negative Ricci of small volume. Wu [**16**] generalized Myers' theorem to the case where Ricci is negative on a set of small diameter (a deep well of small diameter), and Elworthy-Rosenberg gave an extension of Myers' theorem to the case where Ricci is allowed to be a little bit negative on a set of small volume (a shallow well of small volume). Finally, Shen and Wei [**13, 15**] generalized Milnor's theorem, allowing the lower bound on Ricci curvature to be sufficiently small in relation to the diameter and either the volume or first systole of the manifold (a shallow, wide well).

In this paper, we show that all these extensions for compact manifolds can be essentially obtained by placing integral bounds on the negative part of the Ricci curvature (or the part of the Ricci curvature lying below a fixed constant), although in some cases there are technical differences in the hypotheses. From

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