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On the distribution of primes in short intervals

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1. Introduction.

In a recent paper [**P**], the second named author showed that an estimate of the form

$$J_1(N, H) = o(NH^2) \qquad \text{for} \quad H \geqq N^{\theta}, \tag{1}$$

where $0<\theta<1$ and

$$J_1(N, H) = \int_N^{2N} |\psi(x+H)-\psi(x)-H|^2 dx,$$

follows from an estimate of the form

$$\int_N^{2N} |E(x, T)|^2 dx = o\left(\frac{N^3}{T^2 L}\right) \qquad \text{for} \quad T \leqq N^{1-\theta} L. \tag{2}$$

Here $L=\log N$ and $E(x, T)$ denotes the remainder term in the classical explicit formula

$$\psi(x) = x - \sum_{|\gamma|\leq T} \frac{x^{\rho}}{\rho} + E(x, T)$$

where $\rho=\beta+i\gamma$ runs over the non-trivial zeros of the Riemann zeta function. It is well known, see e.g. ch. 17 of Davenport [**D**], that

$$E(x, T) \ll \frac{x \log^2 x}{T}. \tag{3}$$

Since (2) is only a power of L stronger than the estimate which follows from (3), it may appear somewhat surprising that a bound of the form (2) implies a bound of the form (1), for every $0<\theta<1$.

In this paper we give a partial explanation of the above implication. Indeed, in [**K-P**] we obtain the following new form of the explicit formula. Let $0<\varepsilon<1/4$,

$$w(u) = \begin{cases} 1 & 0 \leqq u \leqq \dfrac{1}{2} \\ 2(1-u) & \dfrac{1}{2} \leqq u \leqq 1, \end{cases} \qquad \operatorname{sgn}(u) = \begin{cases} 1 & u>0 \\ 0 & u=0 \\ -1 & u<0 \end{cases}$$