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Groups associated with unitary forms of Kac-Moody algebras

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Introduction.

The groups associated with Kac-Moody algebras, the Kac-Moody groups, were constructed and studied in Peterson-Kac [**9**]. These groups are, so to speak, "infinite dimensional Chevalley groups", and inherit many good properties of finite dimensional Chevalley groups. For example, they have BN-pairs, the corresponding flag varieties are defined, and their Bruhat decompositions are given. Moreover, among the Bruhat cells, the same closure relation holds, in a certain topology, as in the case of finite-dimensional semisimple algebraic groups. (See [**9**] and also Tits [**10**].)

But, it seems to us that this type of construction has some disadvantage for applications to the representation theory. In fact, the exponential map is not defined on the whole Lie algebra, but only for certain generators of algebra (elements of Cartan subalgebra and root vectors of real roots). In other words, there do not exist the 1-parameter subgroups which correspond to root vectors of imaginary roots. (See §1 and §4 for details.) As a consequence, even if we could differentiate a representation of a Kac-Moody group, there should be so much difficulties to examine if the differential gives a representation of its Lie algebra or not.

Motivated by these observations, we attempted to construct groups associated with Kac-Moody algebras in such a way that the exponential map is defined for every element of algebras. This is partially achieved in this paper. We construct such groups corresponding to certain real forms, called unitary forms, of complex Kac-Moody algebras with symmetrizable generalized Cartan matrices (GCM). For finite dimensional complex semisimple Lie algebras, unitary forms are nothing but compact real forms.

We now explain the construction. Let $\mathfrak{g}$ be a complex Kac-Moody algebra with symmetrizable GCM, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{k}$ the unitary form of $\mathfrak{g}$. Let $\Lambda\in\mathfrak{h}^*$ be a dominant integral element and $L(\Lambda)$ the irreducible highest weight module for $\mathfrak{g}$ with highest weight Λ. It was proved in [**6**, Theorem 1] that $L(\Lambda)$ has a canonical pre-Hilbert space structure under which