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Cyclical coincidences of multivalued maps

Dedicated to Professor A. Granas

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0. Introduction.

If $T: X \to 2^Y$ and $U: Y \to 2^X$, $(x, y) \in X \times Y$ is a *coincidence* of T and U if $Tx \ni y$ and $Uy \ni x$. In a recent paper, Browder, [4], has proved the existence of coincidences in a variety of situations. In this paper we shall extend Browder's results to the case of m ($\geqq 2$) spaces. The basic tool that we use is Brouwer's fixed-point theorem for a simplex (though we could equally well use the KKM theorem). We prove in Corollary 3.2 that if, for each $i=0, \cdots, m-1$, X_i is a nonempty convex subset of a topological vector space and $T_i: X_i \to 2^{X_{i+1}}$ has nonempty convex values (with $(m-1)+1$ interpreted as 0) then there exists $(x_0, \cdots, x_{m-1}) \in X_0 \times \cdots \times X_{m-1}$ such that

$$\text{for all} \quad i=0, \cdots, m-1, \qquad T_i x_i \ni x_{i+1}$$

provided that each T_i is either "Browder-Fan" (Definition 1.2) or of "Kakutani type" (Definition 2.3). These definitions will require that some (but possibly not all) of the sets X_i are compact and some (but possibly not all) of the underlying topological vector spaces are locally convex. It is curious that the two kinds of map can be mixed in any order. Corollary 3.2 is a consequence of the main existence theorem, Theorem 3.1, in which we allow some of the maps to satisfy a weaker condition than "Browder-Fan". This weaker condition does not require the vector spaces to be topologized, since it is stated in terms of the "polytopology", which is an intrinsic topology defined on any nonempty convex subset of a vector space (Definition 1.1). The proof of Theorem 3.1 goes by way of two special cases, Theorem 1.4 and Theorem 2.5.

If X is a nonempty compact convex subset of a topological vector space and for all $f \in E'$,

$$Sf = \{x : x \in X,\ f(x) = \max f(X)\}$$

then $S: E' \to 2^X$ has nonempty closed convex values. In Theorem 2.2, we prove a coincidence theorem that involves the map S. This result (which was proved