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The canonical lifting of an ordinary Jacobian variety need not be a Jacobian variety

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We denote by k a perfect field of characteristic p, with $p>0$, and by $A=W(k)$ the ring of infinite Witt vectors over k. Let C_0 be a complete, non-singular curve of genus g over k; we say that C_0 is *ordinary* if its Jacobian variety $\mathrm{Jac}(C_0)$ is an ordinary abelian variety, i.e.

$$\mathrm{Jac}(C_0)[p](\bar{k}) \cong (\boldsymbol{Z}/p)^g, \qquad \text{where } g=\mathrm{genus}(C_0)=\dim(\mathrm{Jac}(C_0)).$$

Let (X_0, λ_0) be a polarized abelian variety and suppose that X_0 is ordinary. By a theorem of Serre and Tate (cf. 1.1) it has a *canonical lifting* $(\mathcal{X}, \lambda)$ to $\mathrm{Spec}(A)$.

We study the following problem (cf. Katz [**4**], p. 138).

PROBLEM. Is the canonical lifting of the Jacobian $(X_0, \lambda_0)=\mathrm{Jac}(C_0)$ of an ordinary curve C_0 again a Jacobian?

Note that if $(\mathcal{X}, \lambda)$ is a polarized abelian variety over $\mathrm{Spec}(B)$, where B is a discrete valuation ring or a field, we say "$(\mathcal{X}, \lambda)$ is a Jacobian" if there exists a field $L\supset B$, and a complete stable curve D over L, such that its canonically polarized generalized Jacobian variety is:

$$\mathrm{Jac}(D)\cong(\mathcal{X}, \lambda)\otimes_B L.$$

Note that the answer to the problem is affirmative if $g\leqq 3$, because by A. Weil for $g=2$ (cf. [**15**], p. 37, Satz 2), and by Oort-Ueno for $g\leqq 3$ (cf. [**10**]), we know that in this case a principally polarized abelian variety is a Jacobian.

In this note we show that in general the answer to the problem is negative (cf. Cor. 2.5 below, also cf. Remark 2.6).

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