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Ergodic affine maps of locally compact groups

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§ 0. Introduction.

Let X be a locally compact group with a left invariant Haar measure μ. Let $f_a = af : X \circlearrowleft$ be a continuous affine map where f is a continuous group automorphism of X and $a \in X$. f_a is said to be *ergodic* under μ if it is measurable and whenever $E \subset X$ is a Borel set such that $f_a(E) = E$ we have either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. The shift map σ of $\boldsymbol{Z}$ is a translation defined on the discrete group $\boldsymbol{Z}$ of integers by $\sigma(n) = n+1$.

Recently N. Aoki [1] has answered the problem of Halmos (p. 29 of [7]) negatively, i.e., if X is a locally compact totally disconnected group which has an ergodic continuous automorphism with respect to a Haar measure μ, then X is compact. For the affine maps, the problem of Halmos remains an open question when X is totally disconnected.

The purpose of this paper is to prove the following:

THEOREM. *Let X be a locally compact group with a left invariant Haar measure μ and $f_a : X \circlearrowleft$ be a continuous affine map. Let $\sigma : \boldsymbol{Z} \circlearrowleft$ be the shift map. If (X, f_a, μ) is ergodic, then either X is compact or (X, f_a) is homeomorphic to $(\boldsymbol{Z}, \sigma)$.*

In N. Aoki's proof, concepts of the pseudo-orbit tracing property and topological mixing for topological dynamics play an important role. We shall apply his techniques for the proof of Theorem.

REMARK 1. Let X, f_a and μ be as in Theorem. If (X, f_a, μ) is ergodic and if X is discrete, either X is compact or (X, f_a) is homeomorphic to $(\boldsymbol{Z}, \sigma)$. Indeed, if X is finite then X is compact. If X is infinite, then $X = \{f_a^n(x) ; n \in \boldsymbol{Z}\}$ for each $x \in X$ by ergodicity of (X, f_a, μ). We define a homeomorphism φ of $\boldsymbol{Z}$ onto X by $\varphi(n) = f_a^n(x)$ $(n \in \boldsymbol{Z})$, and then we get $\varphi \circ \sigma = f_a \circ \varphi$ on $\boldsymbol{Z}$.

For the subclasses of abelian groups and connected groups, the following results are known.

THEOREM A (N. Aoki and Y. Ito [2]). *Let X be a locally compact abelian group with a left invariant Haar measure μ. If on X there exists an affine map*