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Restricted principal values of bounded analytic and harmonic functions

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1. Introduction.

Let f be a function defined on the open unit disc $D=\{z:|z|<1\}$. Let P be a point of the unit circle $C=\{z:|z|=1\}$. A cluster value α achieved by f at P (i.e., $\alpha=\lim f(z_n)$ for some sequence z_n in D with $z_n \rightarrow P$) is called a *principal value of f at P* if it is a cluster value of f along every curve in D terminating at P. A classical theorem of Gross [8] locates a large class of principal values of a bounded analytic function.

THEOREM 1.1 (Gross' Principal Value Theorem). *Let f be a bounded analytic function on D. Suppose α is a cluster value of f along the radius drawn to a point P on C. Suppose α is also an accumulation point of the set of complex numbers which f fails to achieve in D in some neighborhood of P. Then, α is a principal value of f at P.*

This theorem is considered to be quite deep and its most recent proof [6] is commensurately difficult. In contrast, the results in the present paper represent special cases and variations of the Gross theorem and depend on techniques which can be classified as maximum modulus type theorems. For example, one variation of the second condition on α in Theorem 1.1 which is thematic in this paper is that α be a boundary point of the collection of all cluster values of f at P. On the other hand, this stronger hypothesis often allows us to obtain more detailed information about the boundary behavior of f at P. For example, we show that whether or not it is a radial cluster value each such boundary point is a cluster value of f on every curve to P which is more tangential than some corresponding tangential curve. Our techniques are also adequate to completely characterize the one-sided tangential principal values of real bounded

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