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On some additive divisor problems

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§ 1. Introduction.

Let $f(n)$ be a multiplicative function, and let $d(n)$ and $r(n)$ be the number of divisors and the number of representations as a sum of two squares of n, respectively. We consider the problem to find the asymptotic behaviour of the sums

$$\sum_{n\leqq N} d(n)f(n+a), \tag{1.1}$$

$$\text{(as } N\to\infty)$$

$$\sum_{n\leqq N} r(n)f(n+a), \tag{1.2}$$

where a is an arbitrary non-zero integer. The conjugate sums

$$\sum_{n\leqq N} d(n)f(N-n), \tag{1.3}$$

$$\text{(as } N\to\infty)$$

$$\sum_{n\leqq N} r(n)f(N-n), \tag{1.4}$$

where N runs over integers, may also be considered.

These problems have been treated by various authors mainly in the case $f(n)=d_k(n)$ the number of representations of n as a product of k factors. The first general result was obtained by Linnik [**4**], who proved the asymptotic formula for the sum (1.1) in the case of $a=1$ and $f(n)=d_k(n)$ with arbitrary $k\geqq 2$ by appealing to his own powerful 'Dispersion Method'. In his method the work of A. Weil on the Kloosterman sum plays vital part. Later his result was extended by Bredikhin [**2**].

The complexity and difficulty of the dispersion method compelled us to seek a different way, and we have found in [**6**] that the improved large sieve method due to Bombieri [**1**] enables us in some cases to simplify the proof as a whole as well as to dispense with Weil's work. This way of investigation has been further developed and strengthened by Wolke [**10**] [**11**], who has solved the fairly general problem (1.1) in which f is restricted only by the size of $f(n)$ and by the average behaviour of $f(p)$, p a prime. In both Wolke's and our works the main concern is the proof of the analogue for $f(n)$ of the