J. Math. Soc. Japan
Vol. 25, No. 4, 1973

On the coincidence of two Dirichlet series associated with cusp forms of Hecke's "Neben"-type and Hilbert modular forms over a real quadratic field

By Hidehisa NAGANUMA*

(Received May 8, 1972)

1. As the title indicates, the purpose of the present note is to consider a relation, which is analogous to the "decomposition theorem" in the case of L-functions of algebraic number fields, between the Dirichlet series of Hecke type associated with the cusp forms belonging to a Hilbert modular group over a real quadratic field and the series associated with the modular forms of "Neben"-type in Hecke's sense. It may be observed that the problem of this investigation is in the same framework of our previous paper [1] in collaboration with Doi. In fact, the principle of the proof of the result is essentially the same as that of [1]. Let N be a positive integer and ψ_N a character of $(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$ such that $\psi_N(-1)=1$. Put

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid c \equiv 0 \bmod N\right\}.$$

We let $\mathfrak{H}$ denote the upper half complex plane: $\mathfrak{H}=\{\tau \in \boldsymbol{C} \mid \mathrm{Im}(\tau)>0\}$. Let $S_k(\Gamma_0(N), \psi_N)$ be the space of cusp forms of weight k on $\mathfrak{H}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\psi_N(d)(c\tau+d)^k f(\tau) \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{1.1}$$

We shall take N to be *a prime number* q throughout the present investigation and let ψ_q be a non-trivial character of order 2. (The elements of $S_k(\Gamma_0(q), \psi_q)$ are called cusp forms of "Neben"-type after Hecke.) Hereafter we denote the real quadratic field $\boldsymbol{Q}(\sqrt{q})$ by F. We assume that *k is an even positive integer, and the class number of F is one.* Note that $q \equiv 1 \bmod 4$, since $\psi_q(-1)=1$. Let $f(\tau)=\sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$ be an element of $S_k(\Gamma_0(q), \psi_q)$. Suppose that $f(\tau)$ is a common eigen-function of Hecke operators $T(n)$ for all n, and $a_1=1$. Then the Mellin transform of $f(\tau)$ defines a Dirichlet series $L(s, f)$ with the Euler product:

* This research was supported in part by Sakkokai Foundation.