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On certain character groups attached to algebraic groups

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§ 0. Introduction.

This paper is a continuation of my previous papers [9] and [10]. Using the duality theorems of Tate [1], we simplify the results in [9] and [10]. Our main tools are the auxiliary $\mathfrak{g}$-modules defined in [11]. Then our main results become mere applications of the duality theorems of Tate to the fundamental groups of simple algebraic groups. The $\mathfrak{g}(\bar{k}/k)$-module structures of the fundamental groups and their Galois cohomology over an algebraic number field k are already treated in Ono's [6] which is mainly concerned with the relative Tamagawa number of algebraic groups.

Let F be a quasi-split simple algebraic group defined over an algebraic number field k, and Z be the fundamental group of F (in the sense of algebraic groups) which is a finite $\mathfrak{g}$-module. Note that we denote by $\mathfrak{g}$ the Galois group of an algebraic closure $\bar{k}$ of k over k. We denote by F_A the adele group of F over k. It is shown in [9] and [10] that $F_k \cdot [F_A, F_A]$ is closed in F_A, where $[F_A, F_A]$ is the commutator subgroup of F_A, and that the quotient group $A_k(F) = F_A/F_k \cdot [F_A, F_A]$ is a totally disconnected compact group. In this paper, we consider the dual group $\Phi_k(F)$ of $A_k(F)$ in the sense of Pontrjagin, and show that

$$\Phi_k(F) \simeq H^1(\mathfrak{g}, Z'),$$

where $Z' = \mathrm{Hom}(Z, \boldsymbol{G}_m)$ (See Theorem 4). This is our main theorem.

In § 2, we investigate the $\mathfrak{g}$-module structure of the fundamental group Z, using the auxiliary $\mathfrak{g}$-modules defined in (4) and (5). In § 3, we consider their cohomology groups. In § 4, we give an alternative proof of the Hasse principle to the fundamental group Z (Theorem 2) (cf. [6], p. 106-107). In § 5, we prove our main theorems (Theorem 3 and Theorem 4). In § 6, we investigate more explicit structure of $H^1(\mathfrak{g}, Z')$ for some cases. In § 7, we apply our main theorems to calculate the class number of a lattice in its genus.

Some special notations.

We denote by μ_e the group of e-th roots of unity in $\bar{k}$ which has a natural $\mathfrak{g}$-module structure, and by $\boldsymbol{Z}_e$ the cyclic group of order e on which $\mathfrak{g}$ operates