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Fractional powers of operators, III
Negative powers

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This is a continuation of the author's work on fractional powers of operators A in a Banach space X whose resolvent $(\lambda+A)^{-1}$ exists for $\lambda>0$ and satisfies $\|\lambda(\lambda+A)^{-1}\| \leqq M<\infty$, $0<\lambda<\infty$. This part deals with fractional powers A^{α} of exponent α with negative real part, and their relationship with interpolation spaces of X and the range $R(A^{m})$. A unified discussion of mean ergodic theorems is also given, which may be regarded as the theory of A^{-0}.

We use the same notations as in [2] and [3] throughout this paper. In particular, A stands for a closed linear operator in a Banach space X such that $(0, \infty)$ is contained in the resolvent set of $-A$ and that

$$\|\lambda(\lambda+A)^{-1}\| \leqq M, \qquad 0<\lambda<\infty .$$

Such an operator A will be called *non-negative.* The negatives of the infinitesimal generators of bounded continuous semi-groups are non-negative.

When we discussed the basic properties of fractional powers A^{α}, $\alpha \in \boldsymbol{C}$, of non-negative operators A in [2], the following generalization of abelian ergodic theorem played an important role:

$(\lambda+A)^{-1}x$ has the asymptotic expansion

$$(\lambda+A)^{-1}x=\lambda^{-1}x_0-\lambda^{-2}x_1+\cdots+(-1)^{n}\lambda^{-n-1}x_n+o(\lambda^{-n-1}) \quad \text{as } \lambda\to\infty$$

$$(=\lambda^{-1}x_h+x_{-1}-\lambda x_{-2}+\cdots+(-1)^{n-1}\lambda^{n-1}x_{-n}+o(\lambda^{n-1}) \quad \text{as } \lambda\to 0)$$

if and only if $x\in D(A^{n})$ and $A^{n}x\in \overline{D(A)}$ ($x=x_h+A^{n}x_{-n}$ with $x_h\in N(A)$ and $x_{-n}\in D(A^{n})\cap \overline{R(A)}$, respectively).

Also important were the subspaces D^{σ} (and R^{σ}) of X composed of elements x for which the remainder in the above expansion has the order $O(\lambda^{-\sigma-1})$ ($O(\lambda^{\sigma-1})$ and $x_h=0$, respectively).

In view of the above theorem, D^{σ} seems to give an interpolation space. Actually we proved in [3] that D^{σ} coincides with the mean interpolation space $S(\infty, \sigma/m, X; \infty, \sigma/m-1, D(A^{m}))$ of Lions-Peetre [6] if σ is not an integer and m is an integer greater then $\sigma>0$. We also obtained a related characterization of elements x in the interpolation space $D_{p}^{\sigma}=S(p, \sigma/m, X; p, \sigma/m-1, D(A^{m}))$ for $1\leqq p\leqq\infty$ or $p=\infty-$ in terms of $(\lambda+A)^{-m}x$. ($(\lambda+A)^{-m}x$ is more convenient