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# On some double coset decompositions of complex semi-simple Lie groups

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It is well known that a homogeneous space $X$ of a complex semi-simple Lie group $G^c$ by a Borel subgroup $B$ is a compact complex Kähler manifold. A compact form $G_u$ of $G^c$ also operates transitively on $X$ and so $X$ can be considered as a homogeneous space of $G_u$. This manifold was studied profoundly by Borel [1], Borel-Hirzebruch [2], Bott [3], and by Kostant [15]. This manifold is intimately related to the theory of finite dimensional representations of $G_u$ (or of $G^c$). On the other hand, Gelfand and Graev [11] paid attention to that since it is also very important for the theory of unitary representations of a non compact real form $G$ of $G^c$ which is a real semi-simple Lie group. They studied the orbits of $G$ in $X$ when $G$ is of AI-type (or of AIII-type) and, in fact, they showed that one can construct some new irreducible unitary representations of $G$, so-called "discrete series" in their sense, which are realized on some suitable spaces of functions on these orbits [12].

In this paper we shall show that, in general, the space of the double cosets $B\backslash G^c/G$ is finite, i.e. the number of the $G$-orbits in $X$ is finite. In several examples, we shall calculate the number of $G$-orbits. In particular, open orbits in $X$ seem very interesting. These are complex manifolds on which $G$ operates holomorphically, and can be considered as homogeneous spaces of $G$ by its Cartan subgroups.

It seems to be interesting to determine whether they are Stein manifolds or not, and further to study the structure of their cohomologies, because these properties have a close connection with representations of $G$. The method which I have made use of is essentially the same as that of Sul'din in [12].

In § 1, we first prove that the subgroup of all elements are fixed by an anti-linear involutive automorphism of a connected, simply connected complex semi-simple group, which derives easily from the well known Cartan's theorem on symmetric spaces. The decomposition theorem of Bruhat-Chevalley deduces the theorem (Theorem 1):

*Let $G^c$ be a connected complex semi-simple group, $\tilde{\sigma}$ be an anti-linear involutive automorphism of $G^c$, $\Phi$ be the set of all elements $g$ of $G$ which satisfy*