Journal of the Mathematical Society of Japan Vol. 11, No. 3, July, 1959

On some relations concerning the operations P_α and S_α on classes of sets.

By Tadashi OHKUMA

(Received March 3, 1958)
(Revised Dec. 24, 1958)

Introduction.

As extensions of the σ-operation and δ-operation which appear in the theory of usual Borel sets, operations S_α and P_α were already considered in [1], [2] and [3] (cf. Def. 1). Especially in [1] and [2] a condition under which $P_\alpha S_\beta(K)$ is included in $S_\gamma P_\delta(K)$ for any class K of sets is obtained. Referring to these results, we have attempted to study the conditions under which some inequalities or equalities hold between $P_\delta S_\varepsilon$, $P_\alpha S_\beta P_\gamma$, $S_\alpha P_\beta S_\gamma$ etc.

In section 1 several definitions are given. We call the product of operations P_α, S_β etc. a monomial (cf. below Def. 1). In section 2 we shall give a method by means of which the comparison of $P_\delta S_\varepsilon$ with other monomials is fairly simplified and unified. This method is an extension of that used in [1] or [2]. In section 3, a condition for the inequality $S_\alpha P_\beta S_\gamma \leqq P_\delta S_\varepsilon$ or $S_\alpha P_\beta S_\gamma \leqq S_\delta P_\varepsilon$ is obtained. In section 4, we shall first study the condition for the inequality $P_\delta S_\varepsilon \leqq P_\alpha S_\beta P_\gamma$ and next the condition for the equality $P_\delta S_\varepsilon = P_\alpha S_\beta P_\gamma$.

These results are obtained without the generalized continuum hypothesis, but we have not succeeded to give without this hypothesis a condition under which the inequality $P_\delta S_\varepsilon \leqq S_\alpha P_\beta S_\gamma$ holds. Assuming this hypothesis, we shall give a condition for the above inequality in section 5. A condition for the equality $P_\delta S_\varepsilon = S_\alpha P_\beta S_\gamma$ to hold is obtained without the hypothesis.

Throughout this note, the symbol $\pi_\alpha(\beta)$ (cf. Def. 3) plays a main rôle. In section 6, we shall consider the behaviour of the value of $\pi_\alpha(\beta)$, especially we shall give a conditions under which we have $\pi_\alpha(\beta)=\beta$, $\pi_\alpha(\beta)=\beta+1$ or $\pi_\alpha(\beta) \geqq \beta+2$.

§1. Definitions.

1. The following definition of the operation S_α (resp. P_α) is given in [1], [2] and [3].

Definition 1. *Let K be any class of sets, and α an ordinal number. $S_\alpha(K)$ (resp. $P_\alpha(K)$) is the class of all sets which are expressed as the unions (resp.*