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Norm of units of quadratic fields.

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Let P be the rational number field and $\Omega=P(\sqrt{d})$ a real quadratic field, where d is a positive square free integer, different from 2. We denote by ε_0 a fundamental unit of Ω; by ε an arbitrary unit of Ω; by N the absolute norm; and by small Roman letters $a, b, \cdots, m, \cdots$ rational integers.

In this paper, we shall be concerned with the following problem:

"*For what pair of integers d, m does there exist in Ω a ring unit*[1] ε mod. *m with a negative norm*: $N\varepsilon=-1$?"

Dirichlet gave some criteria on the question by means of power residue symbols. More recently it was investigated by A. Scholz, L. Rédei and others. In particular, Rédei [**6**], [**7**] etc.[2], discussed it in detail by using the quadratic residue symbol and the fourth power residue symbol of Dirichlet, and finally Rédei [**9**] solved it completely as a problem related to the ideal class group of quadratic fields. On the other hand, Kuroda [**5**] and Furuta [**1**], [**2**] used the power residue symbol of Dirichlet and a generalized symbol to express the decomposition law of primes in some meta-abelian extensions, and also Tsunekawa [**10**] proved an interesting result concerning our problem. In the present paper, we shall give relationships between the norm of units of real quadratic fields and meta-abelian extensions, from which various results on our problem, in particular some of Rédei's results and Tsunekawa's theorem in a stronger from, can be deduced.

§ 1. Restricted power residue symbol.

Let Δ be an algebraic number field of finite degree, $\mathfrak{p}$ a prime ideal of Δ prime to 2 and α a number of Δ, prime to $\mathfrak{p}$. Then for a non-negative rational integer n the *restricted 2^n-th power residue symbol* $[\alpha/\mathfrak{p}]_n$ is defined as follows[3]. For $n=0$ we set always $[\alpha/\mathfrak{p}]_n=1$. For $n\geqq 1$ $[\alpha/\mathfrak{p}]_n$ is defined only when we have $[\alpha/\mathfrak{p}]_{n-1}=1$, and if this is really the case we set $[\alpha/\mathfrak{p}]_n=(-1)^x$, where $\alpha^{(N\mathfrak{p}^h-1)/2^n}\equiv(-1)^x$ (mod. $\mathfrak{p}$), h being the smallest natural number with $2^n | N\mathfrak{p}^h-1$. For an ideal $\mathfrak{m}$ of Δ prime to both α and 2 with the

1) Namely, a unit ε such that ε is contained in the ring class mod. m.

2) See Rédei [**9**], in which the history and literatures of the subject is stated.

3) See Furuta [**2**]. If Δ containes all the l-th roots of unity for a fixed rational prime l, we shall have analogous results to this § 1 by using l instead of 2.