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# On measures invariant under given homeomorphism group of a uniform space. (A generalization of Haar measure.)

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**Introduction.** Let $\Omega$ be an abstract space. By an "outer measure" $m^*$ in $\Omega$, is meant a non-negative, real valued, countably subadditive set function defined on the class of all subsets of $\Omega$, that is, a set function which satisfies the following conditions:

(1) $m^*(E)\geqq 0$ for every subset $E$ of $\Omega$, $m^*(\theta)=0$ where $\theta$ denotes the null-set.

(2) $E_1\subseteqq E_2$ implies $m^*(E_1)\leqq m^*(E_2)$.

(3) $m^*(\bigcup_{n=1}^{\infty} E_n)\leqq \sum_{n=1}^{\infty} m^*(E_n)$.

A set $E$ is called to be $m^*$-measurable if, for every subset $A$ of $\Omega$,

(4) $m^*(A)=m^*(A\cap E)+m^*(A\cap E^c)$, where $E^c$ denotes the complement of $E$.

It is well known that the class of all $m^*$-measurable subsets of $\Omega$ is a $\sigma$-additive (countably additive) class and $m^*$ is a $\sigma$-additive measure on that class. For measurable subset $E$ we shall write habitually $m(E)$ instead of $m^*(E)$.

If a group $\mathfrak{h}$ of transformations of $\Omega$ is given, then it is natural to consider, as a generalization of Haar measure, an $\mathfrak{h}$-invariant outer measure $m^*$, that is, an outer measure $m^*$ such that

(5) $m^*(\sigma E)=m^*(E)$ for every subset $E\subseteqq\Omega$ and every $\sigma\in\mathfrak{h}$.

From this point of view, the Haar measure can be considered as follows:

(A) Let $\mathfrak{g}$ be a locally compact group. To each element $a\in\mathfrak{g}$ we make correspond a transformation $\varphi_a$ of $\mathfrak{g}$ such that $\varphi_a(x)=ax$ for every $x\in\mathfrak{g}$. And we define $\varphi_a\varphi_b(x)=\varphi_a(\varphi_b(x))$. Then clearly we have $\varphi_a\varphi_b=\varphi_{ab}$. Hence the set $\{\varphi_a : a\in\mathfrak{g}\}$ can be regarded as a transformation group of $\mathfrak{g}$ by defining the group operation as above. We shall denote this transformation group by $\mathfrak{g}_1$. Of course $\mathfrak{g}_1$ is isomorphic with $\mathfrak{g}$ as an abstract group. If we set $\Omega=\mathfrak{g}$ and $\mathfrak{h}=\mathfrak{g}_1$, then our $\mathfrak{h}$-invariant outer measure in $\Omega$ is nothing but a left-invariant Haar (outer) measure in $\mathfrak{g}$.

Let $\mathfrak{g}$ be a locally compact and $\sigma$-compact group and $m^*$ a left-invariant Haar measure. Then we have the following: