Journal of the Mathematical Society of Japan Vol. 10, No. 2, April, 1958

Some remarks on tensor invariants of $O(n)$, $U(n)$, $Sp(n)$.

By Nagayoshi IWAHORI

(Received Jan. 31, 1958)

1. In a paper of Kuiper-Yano [1], tensor invariants of order $\leqq 4$ of special orthogonal groups $SO(n)$ are determined, and the results obtained are applied on the geometry of Riemannian spaces and Finsler spaces.

Using an analogous method on the tensor invariants of the real representations of unitary groups $U(n)$, T. Fukami [2] has obtained corresponding results for $U(n)$, and applied them on hermitian and Kaehlerian spaces.

Now, as we shall show in this note, these problems can be treated more conveniently by intrinsic method than in using tensor components. Thus the results of [1], [2] can be easily generalized for tensor invariants of higher orders, and the cases of groups $O(n), SO(n), U(n), SU(n)$ and $Sp(n)$ can be treated in parallel. In Kuiper-Yano [1], tensor invariants of the subgroup of $SO(n)$ consisting of proper orthogonal transformations which fix a given vector are also determined, however the corresponding problem is not treated in [2]. We shall also show that for the groups $O(n), SO(n), U(n)$, $SU(n)$ and $Sp(n)$, the problem of the determination of tensor invariants of the subgroup consisting of transformations which fix a subspace elementwise is reduced to that of original groups.

2. Let G be a group and (ρ, U) a representation of G on a vector space U. Then

$$U^{\sharp}=U^{\sharp}(G)=\{x\in U;\ \rho(\sigma)x=x \quad \text{for every } \sigma \text{ in } G\}$$

is a subspace of U. An element of $U^{\sharp}$ is called invariant of G in the representation (ρ, U). Now, let G be a group of linear transformations of a vector space V, and U the space of all tensors of type (r, s) over V, i. e.

$$U=V_s^{\ r}=\underbrace{V\otimes\cdots\otimes V}_{r\text{-times}}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{s\text{-times}},$$

where $\otimes$ means the tensor product and V^* means the dual vector space of V. Then, as is well-known, U becomes a representation space of G under the representation ρ defined as follows:

$$\rho(\sigma)(x_1\otimes\cdots\otimes x_r\otimes f_1\otimes\cdots\otimes f_s)=(\sigma x_1)\otimes\cdots\otimes(\sigma x_r)\otimes(\sigma^* f_1)\otimes\cdots\otimes(\sigma^* f_s)$$

where $\sigma\in G$, $x_i\in V$, $f_j\in V^*$ $(i=1,\cdots,r;\ j=1,\cdots,s)$ and σ^* is a linear transformation of V^* given by