Journal of the Mathematical Society of Japan Vol. 6, No. 1, March, 1954

Inner endomorphisms of an associative algebra.

By Akira HATTORI

(Received June 4, 1953)

The inner automorphisms of an algebra are operations of fundamental importance defined on the algebra, where the adjective " inner " implies that these automorphisms are defined by some elements of the algebra by means of a certain canonical procedure which enables one to compute these automorphisms. Not only automorphisms, but homomorphisms of the algebra into itself i. e. endomorphisms of the algebra play some rôles in the structure theory of the algebra. In this connection, an attempt will be made in the following lines to generalize the notion of the inner automorphisms so as to include some endomorphisms which may be called inner in the above sense.

§1. Throughout this paper, A will denote an associative algebra with an identity and of a finite dimension, say n, over a ground field K. Set $K_0=K(X_1,\cdots,X_n)$ and $K_1=K(X_1,\cdots\cdots,X_n, Y_1,\cdots, Y_n)$, where $X_1,\cdots, X_n, Y_1,\cdots, Y_n$ are independent variables over K; an element $f(X_1,\cdots,X_n)$ of K_0, or an element $g(X_1,\cdots,X_n, Y_1,\cdots, Y_n)$ of K_1 will be written in the simplified form $f(X)$ or $g(X, Y)$, respectively. Construct an auxiliary algebra A_1 by extending the ground field K of A to K_1. A K-basis (u_i) of A serves also as a K_1-basis of A_1. Let the multiplication table of (u_i) be $u_i u_j=\sum \gamma_{ijk}u_k$, $\gamma_{ijk}\in K$. If $a=\sum f_i u_i$, $f_i\in K_1$, is an element of A_1, we shall denote by $M(a, u)$ the n-rowed matrix $(\sum_i f_i \gamma_{ijk})_{jk}$. Then a is regular if and only if $\det M(a,u)\neq 0$. Since $\det M(1, u)$ does not vanish certainly, we have also $\det M(S, u)\neq 0$ for the general element $S=\sum X_i u_i$, which has therefore an inverse element. Put

$$(\sum X_i u_i)(\sum Y_i u_i)(\sum X_i u_i)^{-1}=\sum Y_i' u_i\,, \tag{1}$$

where $Y_i'=Y_i'(X, Y)$ is linear in Y_i, and we can write

$$Y_i'=\sum R_{ij}(X)Y_j \qquad \text{with} \qquad R_{ij}(X)\in K_0\,. \tag{2}$$