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A Proof of Schauder's Theorem

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1. Introduction. The purpose of this note is to give a simple proof to the following theorem of J. Schauder[1]: A bounded linear operator T defined on a Banach space X is completely continuous if and only if the adjoint operator T^* of T defined on the conjugate space X^* of X is completely continuous. We shall give a formulation of Schauder's theorem (Theorem 2) in which X and X^* (and hence T and T^*) appear as a dual pair. (It is to be observed that X^* has no need to be the conjugate space of X in Theorem 2). Since T and T^* play equivelent roles in our formulation, the "if" part of the theorem is an equivalent proposition to the "only if" part.

Our proof of Schauder's theorem is based on the following well-known theorem of G. Arzelà: A uniformly bounded, equi-continuous family $F=\{f(x)\}$ of real-valued continuous functions $f(x)$ defined on a totally bounded metric space X is totally bounded with respect to the metric

$$d(f_1, f_2)=\sup_{x\in X} |f_1(x)-f_2(x)|. \tag{1}$$

We shall give a formulation of a special case of Arzela's theorem (Theorem 1) in which X and F play equivalent roles so that the total boundedness of X is also necessary for the total boundedness of F. The notion of totally bounded functions introduced in section 2 will be helpful in making arguments simpler.

2. Totally bounded functions. Let $X=\{x\}$, $Y=\{y\}$ be two sets. Let $f(x,y)$ be a bounded real-valued function defined for all $x\in X$ and for all $y\in Y$.

Lemma 1. The following three conditions are mutually equivalent: (i) *for any* $\varepsilon>0$ *there exists a decomposition* $X=\bigcup_{i=1}^{m}A_i$ *of* X *into a finite number of subsets* A_i, $i=1, \cdots, m$, *such that*

$$|f(x_1, y)-f(x_2, y)|<\varepsilon \tag{2}$$

for all $x_1, x_2\in A_i$ *(same* i*),* $i=1, \cdots, m$, *and for all* $y\in Y$. (ii) *for any* $\varepsilon>0$ *there exists a decomposition* $Y=\bigcup_{j=1}^{n}B_j$ *of* Y *into a finite number of subsets* B_j, $j=1, \cdots, n$, *such that*

$$|f(x, y_1)-f(x, y_2)|<\varepsilon \tag{3}$$

for all $x\in X$ *and for all* $y_1, y_2\in B_j$ *(same* j*),* $j=1, \cdots, n$. (iii) *for any* $\varepsilon>0$ *there exist decompositions* $X=\bigcup_{i=1}^{m}A^i$, $Y=\bigcup_{j=1}^{n}B_j$ *of* X *and* Y *into a finite*