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A remark on Schottky's theorem.

MASATSUGU TSUJI.

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Let $f(z)=a_0+a_1 z+\ldots\ldots$ be regular for $|z|<1$ and $f(z)\neq 0, \neq 1$, then by the precise form of Schottky's theorem due to Bohr and Landau,[1)]

$$|f(z)|\leqq \exp \frac{D\log(|a_0|+2)}{1-r} \quad (|z|=r<1), \quad (1)$$

where D is a numerical constant.
Since from (1), $\{f(z)\}$ forms a normal family, we can easily prove that if $a_0\to 0$, then $f(z)\to 0$ uniformly in the wider sense in $|z|<1$. But the right hand side of (1) does not tend to zero for $a_0\to 0$. Hence it is desirable to obtain a majorant of $|f(z)|$, which tends to zero for $a_0\to 0$. We will now prove the following

Theorem. $|f(z)|\leqq \exp \left(A \log|a_0|\cdot(1-r)+B\dfrac{\log (|a_0|+2)}{1-r}\right)$,

where A, B are positive numerical constants.

Proof. Let D_0 be the domain on $\zeta=x+iy$-plane, such that $o<x<1$, $y>0$, $|\zeta-\frac{1}{2}|>\frac{1}{2}$ and we map D_0 on the upper half w-plane by $w=\lambda(\zeta)$, such that $\lambda(o)=o$, $\lambda(1)=1$, $\lambda(\infty)=\infty$.

Then $\lambda(\zeta)$ is automorphic with respect to a modular group G, whose fundamental domain is $\varDelta=D_0+D_0'$, where D_0' is the image of D_0 with respect to the imaginary axis.
As well known, $w=\lambda(\zeta)$ has a simple pole at $t=o$, where $t=e^{\pi i\zeta}$, so that if $y\geqq\eta$ (>1),

$$e^{\frac{\pi}{2}y}\leqq|w|\leqq e^{2\pi y} \quad (y\geqq\eta), \quad \text{or} \quad (2)$$

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1) H. Bohr und E. Landau: Über das Verhalten von $\zeta(s)$ und $\zeta_\kappa(s)$ in der Nähe der Geraden $\sigma=1$. (Göttinger Nachr. 1910).