On the growing up problem for semilinear heat equations

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§ 1. Introduction.

We will consider the Cauchy problem for the semilinear heat equation

\[ \frac{\partial u}{\partial t} = \Delta u + f(u), \quad t > 0, \quad x \in \mathbb{R}^d. \]

with the initial condition $u(0, x) = a(x)$. It is assumed that the function $f$ is defined, non-negative and locally Lipschitz continuous in $[0, \infty)$. If the initial value $a(x)$ is a bounded non-negative continuous function in $\mathbb{R}^d$, not vanishing identically, then it is well-known that there exists a positive local solution $u(t, x)$ of (1.1); more precisely, there exist positive $T \leq \infty$ and $u(t, x)$ satisfying the following conditions (i), (ii) and (iii).

(i) $u(t, x)$ is defined on $[0, T) \times \mathbb{R}^d$, strictly positive in $(0, T) \times \mathbb{R}^d$ and $u(0, x) = a(x)$.
(ii) For any $T' < T$, $u(t, x)$ is bounded and continuous on $[0, T') \times \mathbb{R}^d$.
(iii) $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j} (1 \leq i, j \leq d)$ exist in $(0, T) \times \mathbb{R}^d$ and $u(t, x)$ satisfies (1.1) in the classical sense.

If $T_\infty = T_\infty(a, f)$ denotes the supremum of all $T$ satisfying the above three conditions, then the existence of global solution is the case $T_\infty = \infty$, and in the general situation ($T_\infty \leq \infty$) the unique existence assertion amounts to say that there exists a unique solution $u(t, x)$ of (1.1) up to $T_\infty$ satisfying the above three conditions with $T = T_\infty$. In this paper, such a solution is called simply a positive solution of (1.1), and is denoted by $u(t, x; a, f)$ when we want to elucidate the initial value $a(x)$ and the nonlinear term $f(u)$. A positive solution of (1.1) is said to blow up in a finite time and the corresponding $T_\infty$ is called the blowing-up time of the solution, provided that $T_\infty < \infty$. A global positive solution $u(t, x)$ of (1.1) is said to grow up to infinity, if for each positive constant $M$ and each compact set $K$ in $\mathbb{R}^d$ there exists $T < \infty$ such that $t > T$ and $x \in K$ imply $u(t, x) > M$.

The purpose of this paper is to investigate the following problem: How