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Isomorphisms of Galois groups

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Let Q be the field of the rational numbers. Let Ω be a normal algebraic extension of Q such that Ω has no abelian extension. Let G be the Galois group of Ω over Q. Neukirch [**4**, **5**] has shown that every open normal subgroup of G is a characteristic subgroup, and has proposed a problem whether every automorphism of G is inner. In this paper this problem is solved affirmatively, i. e., we prove

THEOREM. *Let G_1 and G_2 be open subgroups of G, and let $\sigma: G_1 \rightarrow G_2$ be a topological isomorphism. Then σ can be extended to an inner automorphism of G.*

Kanno [**2**] and Komatsu [**3**] gave partial results which suggested this problem is affirmative. The author first proved this theorem in the case Ω is the algebraic closure or the solvable closure of Q, because Neukirch has stated his theorems in these cases. Ikeda [**1**] solved Neukirch's problem independently almost at the same time. Iwasawa then remarked that his methods are also applicable to the proof of our theorem. Iwasawa also noticed that Neukirch's theorems are valid for every Ω as above. We state our theorem in the above form after his suggestion. The author expresses his hearty thanks to Professors Iwasawa, Kanno and Komatsu.

Let N be any open normal subgroup of G contained in G_1 and G_2. We put $H=G/N$, $H_1=G_1/N$ and $H_2=G_2/N$. Neukirch's theorems (or rather his Satz 12 in [**4**] and Satz 7 in [**5**]) show that $\sigma(N)=N$. Hence σ induces an isomorphism $\sigma_N: H_1 \rightarrow H_2$.

LEMMA 1. *If σ_N can be extended to an inner automorphism of H for every open normal subgroup N, σ can be extended to an inner automorphism of G.*

PROOF. Let g be any element of G and let g_N be an inner automorphism of H induced from g. If we put

$$I_N = \{g \in G \mid \sigma_N = g_N \quad \text{on } H_1\},$$

I_N is non-empty for every N by our assumption. As I_N is a union of some cosets of N, it is non-empty compact set. Let $N_1, \cdots, N_m$ be any open normal subgroups. Then $I_{N_1} \cap \cdots \cap I_{N_m}$ is non-empty because it includes $I_{N_1 \cap \cdots \cap N_m}$.