Certain conditions for a Riemannian manifold to be isometric with a sphere

Dedicated to Professor Kentaro Yano on his fiftieth birthday

By Morio Obata

(Received Feb. 24, 1962)  
(Revised Mar. 31, 1962)

Introduction.

In this paper, by a Riemannian manifold we always mean a connected $C^\infty$-manifold of dimension $n \geqq 2$ with a positive definite $C^\infty$-Riemannian metric. A transformation of a Riemannian manifold is said to be conformal if it preserves the angle defined by the Riemannian metric. Evidently a conformal transformation with respect to one Riemannian metric is also conformal with respect to one conformally related to the original one. It is known [5] that if a manifold is compact and of dimension greater than two, every Riemannian metric on it can be conformally deformed into one with constant scalar curvature. Thus as far as a conformal transformation is concerned, we may assume the constancy of the scalar curvature in the above case. Then the scalar curvature, which is constant, is preserved by the transformation. This property seems to help the study of conformal transformations. Indeed, a conformal transformation between two compact Riemannian manifolds of non-positive scalar curvature, not necessarily constant, is isometric if and only if it carries one scalar curvature into another [4]. Thus a compact Riemannian manifold of constant scalar curvature admits a conformal transformation, which is not isometric, only if the scalar curvature is positive. Therefore we may restrict our consideration to Riemannian manifolds of positive constant scalar curvature as far as a conformal transformation is concerned, provided that the manifold is compact and of dimension $> 2$.

Furthermore, if a Riemannian manifold of constant scalar curvature $k$ admits an infinitesimal conformal transformation $u$ with $\varepsilon_u g = 2\phi g$, where $g$ is the Riemannian metric, $\varepsilon_u$ the operation of Lie derivatives corresponding to $u$ and $\phi$ a function, then $\phi$ satisfies the equation $\Delta \phi = nk\phi$, where $k = K/n(n-1)$, $K$ being the contracted curvature scalar (See for example [2, 6]). The existence of such a function might give some informations about the topological structure of the Riemannian manifold. Indeed we are going to show that a compact Einstein manifold of constant scalar curvature $k$ admits a non-constant