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PROJECTIVE MODULES OVER NON-COMMUTATIVE SEMILOCAL RINGS

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An associative ring R with identity is called a *semilocal* ring in case it is semisimple (artinian) modulo its radical $J = J(R)$. A *semiperfect* ring is a semilocal ring in which idempotents lift modulo J. The structure of projective modules over semiperfect rings has been well determined (see [15], [18], [21]). However, though semilocal rings that are not semiperfect occur naturally enough (see Proposition 4 and Example 5) and have enjoyed recent attention in the literature (e.g., in [1], [4], [8], [10]), except when they are commutative (see [11]), the structure of their projective modules is largely unknown. Here we show that if all non-zero projective left modules over a semilocal ring are generators (e.g., if R/J is simple artinian) then they are direct sums of a fixed idempotent-generated principal left ideal (Theorem 1); that if a finite direct sum of copies of each simple module has a projective cover then every indecomposable projective module is finitely generated (Theorem 6 and Corollary 8); and that semilocal rings have only finitely many finitely generated indecomposable projective modules (Theorem 9).

We use notation and results on projective covers contained in [2].

A ring is called (left) p-connected by Bass [3] in case each of its projective left modules is a generator. (We use the notion $M^{(c)}$ to denote, for a cardinal number c and a module M, a direct sum of c copies of M; so ${}_RG$ is a generator in case there is an epimorphism $G^{(c)} \to {}_RR \to 0$.) Hinohara [11] proved that a commutative semilocal ring in which 1 is a primitive idempotent has all its projective modules free. Akasaki [1] generalized this and Kaplansky's well-known theorem [14] on local rings by proving that projective left R-modules are free if R is a p-connected semilocal ring in which each maximal left ideal is two-sided. Our first result eliminates this vestige of commutativity.

1. THEOREM. *If R is a p-connected semilocal ring then there exists a primitive idempotent $e \in R$ such that every projective left R-module is isomorphic to a direct sum of copies of Re.*

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