NOTES ON FOURIER ANALYSIS (XL): ON THE ABSOLUTE SUMMABILITY OF THE FOURIER SERIES

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1. Let $\{\lambda_n\}$ be a positive and increasing sequence, and let

$$R(\omega) = \omega^{-1} \sum_{\lambda_n < \omega} (\omega - \lambda_n) a_n \tag{1.1}$$

be the $(R, \lambda_n, 1)$-mean of the series $\sum a_n$. If $R(\omega)$ is of bounded variation in the interval (λ_1, ∞), that is

$$\int_{\lambda_1}^{\infty} |dR(\omega)| = \int_{\lambda_1}^{\infty} \omega^{-2} \Big| \sum_{\lambda_n < \omega} \lambda_n a_n \Big| d\omega < \infty,$$

then $\sum a_n$ is said to be absolutely $(R, \lambda_n, 1)$-summable, or simply $|R, \lambda_n, 1|$-summable.

Let $f(t)$ be an L-integrable function in the interval $(0, 2\pi)$, and its Fourier series be

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

For the absolute summability of the Fourier series, following theorems are known:

THEOREM A. [1] *If for any* $\beta > 0$

$$\varphi(t)(\log t^{-1})^{\beta} = O(1) \quad (t \to 0),$$

then the Fourier serier of $f(t)$ *is summable* $|R, \log n, 1|$ *at* $t = x$, *where*

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

THEOREM B. [2] *If* $\varphi(t)$ *is of bounded variation in* $(0, \pi)$, *then the Fourier series of* $f(t)$ *is summable* $|R, n, \varepsilon|$ *at* $t = x$, *where* $\varepsilon > 0$.

THEOREM C. [3] *If* $\varphi(t) \log 1/t$ *is of bouuded variation in* $(0, \pi)$ *then the Fourier series of* $f(t)$ *is summable* $|R, \exp(n^{\alpha}), 1|$ *at* $t = x$, *where* $0 < \alpha < 1$.

In this paper we consider the summability $|R, \exp((\log n)^{\alpha}), 1|$, where $\alpha > 0$, and prove the following theorems:

THEOREM 1. *If* $\varphi(t)(\log 1/t)^{\beta} = O(1)$, *then the Fourier series of* $f(t)$ *is summable* $|R, \lambda_n, 1|$ *at* $t = x$, *where*

$$\lambda_n = \exp((\log n)^{\alpha}), \quad 0 < \alpha < \beta \text{ and } \alpha < 1.$$

THEOREM 2. *If* $\varphi(t)(\log\log 1/t)^{\beta} = O(1)$, *then the Fourier series of* $f(t)$ *is summable* $|R, \log n, 1|$, *where* $\beta > 1$.