ON A REGULAR FUNCTION WHICH IS OF CONSTANT ABSOLUTE VALUE ON THE BOUNDARY OF AN INFINITE DOMAIN

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1. A metric in a circle. First we will introduce a metric in $|w| \leqq R$ as follows. We define the distance $(w, 0)$ of a point w $(|w| \leqq R)$ from $w = 0$ by

$$(1) \qquad (w, 0) = \frac{2R|w|}{R^2 + |w|^2}, \quad (0 \leqq (w, 0) \leqq 1).$$

Let

$$(2) \qquad U_a(w) = \frac{R^2(w-a)}{R^2 - \bar{a}w} \qquad (|a| < R)$$

be a linear transformation, which transforms $|w| < R$ into itself, such that $U_a(a) = 0$. We define the distance (a, b) of any two points a, b in $|w| < R$ by

$$(3) \qquad (a, b) = (U_a(b), 0) = \frac{2R|U_a(b)|}{R^2 + |U_a(b)|^2} = \frac{2R\left|\dfrac{b-a}{R^2 - \bar{a}b}\right|}{1 + R^2\left|\dfrac{b-a}{R^2 - \bar{a}b}\right|^2},$$

so that

$$(4) \qquad (a, b) = (b, a).$$

It is easily seen that for any linear transformation $U(w)$, which transforms $|w| < R$ into itself,

$$(5) \qquad (U(a), U(b)) = (a, b)$$

and a circle $(w, a) = \rho$ in our metric is an ordinary circle and the locus of points, which are equidistant from two given points is a circle, which cuts $|w| = R$ orthogonally.

In our metric, the triangle inequality

$$(6) \qquad (a, c) < (a, b) + (b, c)$$

holds.

PROOF. We may assume that $R = 1$ and $a = 0$, $0 < b < 1$ by (5), so that it suffices to prove:

$$(7) \qquad \frac{|c|}{1 + |c|^2} < \frac{b}{1 + b^2} + \frac{\left|\dfrac{b-c}{1-bc}\right|}{1 + \left|\dfrac{b-c}{1-bc}\right|^2}.$$