GROUP ALGEBRAS IN THE LARGE

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1. Introduction. Let G be a unimodular locally compact group. We form the Hilbert space $L_2(G)$, and the left and right regular representations of G on $L_2(G)$. We shall denote the weakly closed algebras generated by these representations by W and W'. It is known [10] that W and W' are the full commuting algebras of each other.

It is a question of some importance to determine how properties of G are reflected in properties of W. The pioneering investigations of Murray and von Neumann in [8] have been followed by several interesting contributions by Mautner [5], [6], [7]. Mautner's results are phrased in terms of von Neumann's decomposition theory [9].

In this paper we shall prove several theorems about the structure of W "in the large". There are several advantages in this change of point of view. Measure-theoretic questions and "almost everywhere" difficulties disappear; separability becomes irrelevant; and above all, the proofs become simpler.

If the properties of W "in the small" are desired for their own sake, then, in the author's opinion, these are best obtained by using a dictionary for translating properties back and forth. Such a dictionary may be expected in the near future.

2. Definitions. We shall use the terminology introduced in [3], but we collect the definitions for the reader's convenience.

Let W be a weakly closed self-adjoint algebra of operators on a Hilbert space. A non-zero projection e is *abelian* if eWe is commutative; e is *finite* if left and right inverses in eWe coincide. We say that W is finite if its unit element is finite; W is of type I if every direct summand contains an abelian projection; W is of type II if there are no abelian projections and every direct summand contains a finite projection; W is of type III if all projections are infinite; W is of type II_1 if it is finite and of type II. By [3, Th. 4.6] W is uniquely a direct sum of algebras of types I, II and III.

Suppose W is finite and of type I. The structure of W is implicitly given in [3] and may be described as follows: W is a direct sum of algebras W_n $(n = 1, 2, \cdots)$, where W_n is an n by n total matrix algebra over a commutative algebra.

We shall now specialize to the case where W, W' are the group algebras of G. There are then two further known facts. (1) W has no part of type III. (2) W and W' decompose in the same manner; more precisely, if h is any projection in the center of W (which is also the center of W')