NOTE ON DIRICHLET SERIES (III)
ON THE SINGULARITIES OF DIRICHLET SERIES (III)

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9. Theorem V. In this present Note, we shall generalize the Fundamental Theorem 1 established in the previous Note [1][1]. Let us put

$$(1.1)\qquad F(s)=\sum_{n=1}^{\infty}a_n\exp(-\lambda_n s)\qquad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\ldots<\lambda_n\to+\infty).$$

We shall begin with

DEFINITION IV. The subsequence $\{\lambda_{n_k}\}$ $(k=1,2,\ldots)$ is called the normal subsequence of density 0, provided that

$$\text{(a)}\qquad \lim_{k\to\infty}k/\lambda_{n_k}=0,$$

$$\text{(b)}\qquad \varliminf_{k\to\infty}(\lambda_{n_k}-\lambda_{n_{k-1}})>0,\ \varliminf_{\substack{k,n\to\infty\\ n\neq n_k}}|\lambda_{n_k}-\lambda_n|>0.$$

Then, the Fundamental Theorem 1 is generalized in a following manner.

THEOREM V. *Let* (1.1) *be simply convergent for* $\sigma>0$. *Then* $s=0$ *is the singular point for* (1.1), *provided that there exist two sequences* $\{x_k\}$ $(0<x_k\uparrow\infty)$, $\{\gamma_k\}$ (γ_k: *real*) *such that, for a normal subsequence* $\{\lambda_{n_k}\}$ *of density* 0,

$$\text{(a)}\qquad \varlimsup_{k\to\infty}1/x_k\cdot\log\left|\sum_{\substack{[x_k]\leqq\lambda_n<x_k\\ \lambda_n\in\{\lambda_{n_k}\}}}\Re(a_n\exp(-i\gamma_k))\right|=0,$$

(b) $\lim_{k\to\infty}\sigma_k/[x_k]=0$, *where* σ_k: *the number of sign-changes of* $\Re(a_n\exp(-i\gamma_k))$, $\lambda_n\,\overline{\in}\,\{\lambda_{n_k}\}$, $\lambda_n\in I_k[[x_k](1-\omega),\ [x_k](1+\omega)]$ $(0<\omega<1)$,

(c) *the sequence* $\Re(a_n\exp(-i\gamma_k))$ $(\lambda_n\,\overline{\in}\,\{\lambda_{n_k}\},\ \lambda_n\in I_k)$ *has the normal sign-change in* $\{I_k\}$ $(k=1,2,\ldots\ldots)$.

By virtue of this theorem, we get

COROLLARY VIII. *If the hypothesis of all theorems established in the previous Note* (I-II) *are satisfied, except for a normal subsequence* $\{\lambda_{n_k}\}$ *of density* 0, *then these theorems are also valid.*

For example, from Corollary 8 we get

COROLLARY IX (S. Izumi, [2]). *Let* (1.1) *be simple convergent for* $\sigma>0$. *If* $|\arg(a_n)|\leqq\theta<\pi/2$, *except for the normal subsequence* $\{\lambda_{n_k}\}$ *of density* 0, *then* $s=0$ *is singular for* (1.1).

10. Lemma. For its proof, we need next Lemma.

1) Vide references placed at the end.