ON THE REPRESENTATIONS OF POSITIVE DEFINITE FUNCTIONS AND STATIONARY FUNCTIONS ON A TOPOLOGICAL GROUP

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(Received October 19, 1951)

1. Introduction. Let G be a (not necessarily locally compact) topological group. A continuous (complex-valued) function $f(g)$ is called *positive definite* provided that

$$\sum_{i,j=1}^{n} f(g_i g_j^{-1})\alpha_i\overline{\alpha_j} \geqq 0$$

for any set of complex numbers $\{\alpha_i\}$ and elements $\{g_i\}$ of G. A strongly continuous function $x(g)$, being defined on G and having its values in a Hilbert space H, is called, after A. Khintchine [7], *stationary* provided that $(x(g), x(h))$ depends only on gh^{-1}.

A closed connection between positive definite functions and stationary functions is firstly pointed out by K. Fan [2], when G is the (discrete) additive group of integers. He shows that for a positive definite sequence $\{\alpha_n\}$ there exists a stationary sequence $\{x_n\}$ with $\alpha_{n-m} = (x_n, x_m)$. (Converse is naturally obvious.) Establishing this, he developed the theory of positive definite sequences parallel with that of stationary sequences, and proved several theorems without use of the Herglotz Theorem of integral representation.

On the other hand, it is known that I. Gelfand and D. Raikov [3] established the natural one-to-one correspondence between positive difinite functions and unitary representations of locally compact group, that is, for a given continuous positive definite function $f(g)$ there exists a unitary (strongly continuous) representation $U(g)$ with $f(g) = (xU(g), x)$ where x is a suitable element of the representation space. (Although Gelfand-Raikov's paper is not available to the authors, this result is reproduced in R. Godement [4] and H. Yosizawa [9]). Hence, putting $x(g) = x(1)U(g)$, $x(g)$ becomes a stationary function on G, and so Fan's theorem is generalized onto locally compact groups as follows: For any positive definite function $f(g)$ on G there is a stationary function $x(g)$ on G with $f(gh^{-1}) = (x(g), x(h))$.

It seems, therefore, Fan's method of the proof is applicable to this generalization. As it is seen in the below, this is done in § 2, with a few modification. Moreover, it generalized Gelfand-Raikov's Theorem without local compactness (Theorem 1). However, it is not so unexpected. Let G be a topological group (with or without local compactness) and $f(g)$ be a continuous positive definite function on G. Considering G as a discrete group,