ON F.RIESZ' FUNDAMENTAL THEOREM ON SUBHARMONIC FUNCTIONS

MASATSUGU TSUJI

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1. In this paper, I shall prove simply the following F. Riesz' fundamental theorem on subharmonic functions.[1]

THEOREM 1. *Let $u(z)$ be a subharmonic function in a domain D on the z-plane, then there exists a positive mass-distribution $\mu(e)$ defined for Borel sets e in D, such that for any bounded domain $D_1 \subset D$, which is contained in D with its boundary,*

$$u(z) = v(z) - \int_{D_1} \log \frac{1}{|z-a|}\, d\mu(a) \quad (z \in D_1),$$

where $v(z)$ is harmonic in D_1 and such $\mu(e)$ is unique.

The main idea of the proof is as follows.

Let z be any point of D and a disc $\Delta(\rho, z): |\zeta - z| < \rho$ be contained in D and put

$$L(r, z: u) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta \quad (0 < r < \rho). \tag{1}$$

Then $L(r, z: u)$ is an increasing convex function of $\log r$,[2] so that $r\,dL(r, z{:}u)/dr \geqq 0$ exists, except at most a countable number of values of r. We call such a disc $\Delta(r, z)$ a non-exceptional disc. We define the mass μ contained in a non-exceptional disc by

$$\mu(\Delta(r, z)) = \frac{r\, dL(r, z: u)}{dr} \geqq 0. \tag{2}$$

Let e be any set in D. We cover e by at most a countable number of non-exceptional discs $\Delta(r_\nu, z_\nu)$ and put

$$\mu^*(e) = \inf \sum_\nu \mu(\Delta(r_\nu, z_\nu)). \tag{3}$$

Then $\mu^*(e)$ is the Carathéodory's outer measure of e, which is an additive set function for Borel sets e. The main difficulty of the proof is prove that for a non-exceptional disc

$$\mu^*(\Delta(r, z)) = \mu(\Delta(r, z)) \quad \text{(Lemma 3)}.$$

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1) F. RIESZ, Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel II, Acta Math., 54(1930). G. C. EVANS, On potentials of positive mass I, Trans. Amer. Math. Soc., 37(1938). T. RADÓ, Subharmonic functions, Berlin (1937).

2) Rado's book, p. 8.