GENERATION OF A STRONGLY CONTINUOUS SEMI-GROUP OPERATORS

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1. Let $\{T(\xi); 0 \leqq \xi < \infty\}$ be a one-parameter semi-group satisfying the following conditions:

(i) $T(\xi)$ is a bounded linear transformation from a complex (B)-space E into itself.

(ii) $T(\xi + \eta) = T(\xi)\cdot T(\eta)$, $T(0) = I$ (the identity transformation).

(iii) $T(\xi)$ tends to I strongly as $\xi \to 0$, but not uniformly.

We define $Ax = \lim_{h\to 0} \frac{1}{h}[T(h) - I]x$ whenever the limit exists and the set of elements x for which Ax exists, will be denoted by $D[A]$.

Let us put

$$R(\lambda; A)x = -\int_0^\infty e^{-\lambda\xi} T(\xi)x d\xi, \quad \lambda = \sigma + i\tau,$$

for all $x \in E$, then the representation holds at least for all λ such that $R(\lambda) = \sigma > \log M$, where $M = \sup_{0\leqq\xi\leqq 1} \| T(\xi) \|$.

We shall consider a problem of E. Hille [1][1] which is stated as follows:

What properties should an operator A possess in order that it be the infinitesimal generator of a strongly continuous semi-group $\{T(\xi); 0 \leqq \xi < \infty\}$ of bounded linear transformations from a complex (B)-space into itself?

A theorem of E. Hille [1, Theorem 12.2.1] reads as follows.

THEOREM H. *Let A be a closed linear unbounded operator on E into itself whose domain is dense in E. Let the spectrum of A is located in $R(\lambda) = \sigma \leqq 0$ and suppose that*

$$\| R(\sigma + i\tau; A) \| \leqq \frac{1}{\sigma} + \frac{\beta}{\sigma^2}, \quad \sigma > 0, \tag{1}$$

where β is a fixed constant, $\beta \geqq 0$. Then there exists a semi-group $\{T(\xi); 0 \leqq \xi < \infty\}$ such that $T(\xi)$ satisfies the conditions (i)-(iii), $\| T(\xi) \| \leqq e^{\beta\xi}$ *for all $\xi > 0$ and that A is its infinitesimal generator.*

The behavior of the norm, $\| T(\xi) \| \leqq e^{\beta\xi}$, is by no means implied by the conditions (i)-(iii) as may be seen later (Example 1). Therefore this theorem is not the perfect solution of the above problem.

In this paper we shall give a necessary and sufficient condition that the

1) Numbers in brackets refer to the bibliography at the the end of the paper.