ON THE DIRECT-PRODUCT OF OPERATOR ALGEBRAS II

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1. Introduction. R. Schatten-J. von Neumann [4] introduced the idea of direct-product of Banach spaces, and the author modified this considerations to C^*-algebras in the previous paper [7], and defined the direct-product of C^*-algebras.

Let A_1 and A_2 be any C^*-algebras with unit, and following R. Schatten-J. von Neumann. construct $A_1 \times A_2$ as the set of all expressions $\sum x_i \times y_i$ with the equivalence relation $\cong$, as A_1, A_2 to be Banach spaces; and finally define the multiplication, involution and norm of expressions as follows:

$$\text{product:} \qquad \Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{j=1}^{m} s_j \times t_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i s_j \times y_i t_j,$$

$$\text{involution:} \qquad \Big(\sum_{i=1}^{n} x_i \times y_i\Big)^* = \sum_{i=1}^{n} x_i^* \times y_i^*,$$

$$\text{norm:} \qquad \alpha\Big(\sum_{i=1}^{n} x_i \times y_i\Big) = \sup\left[\Phi\Big(\Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{i=1}^{n} x_i \times y_i\Big)^*\Big)^{1/2} : \Phi \in \mathfrak{S}\right],$$

where $\mathfrak{S}$ denotes the set of positive type functional Φ's such that

$$\Phi\Big(\sum_{j=1}^{m} s_j \times t_j\Big) = \frac{\varphi \times \psi\Big(\Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{j=1}^{m} s_j \times t_j\Big)\Big(\sum_{i=1}^{n} x_i \times y_i\Big)^*\Big)}{\varphi \times \psi\Big(\Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{i=1}^{n} x_i \times y_i\Big)^*\Big)}$$

φ and ψ are pure states on A_1 and A_2 respectively, and $\sum_{i=1} x_i \times y_i$ is an arbitrary element of $A_1 \times A_2$; then α becomes a cross-norm on $A_1 \times A_2$ and $A_1 \times A_2$, is a non-complete C^*-algebra [7].

Now, let A_1 and A_2 be C^*-algebras on the Hilbert spaces H_1 and H_2 respectively. Then by [3, 4], $\sum_{i=1}^{n} x_i \times y_i$ can be considered as bounded operator on $H = H_1 \times_\sigma H_2$. In this paper, we consider the relation between C^*-algebra generated by $\sum_{i=1}^{n} x_i \times y_i$ as operator on H with operator bound as norm, and direct-product $A_1 \times_\alpha A_2$ (§2); and we give more detailed discussions in the case where A_i are C^*-algebras of completely continuous operators on H_i (§3); and finally in §4 we prove some algebraic properties of $A_1 \times_\alpha A_2$.